ON FREE Γ-SEMIGROUPS

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Abstract

In this paper we give a construction of free Γ-semigroups using the UMP. We describe some of their properties and finally, we give some results about their presentations.

Keywords: free Γ-semigroup, UMP, Γ-group, presentation.

1 Introduction

As P.A.Grillet has pointed out…"Describing semigroups is a formidable task. Semigroups are among the most numerous objects in mathematics, and also among the most complex…" A semigroup is an algebraic structure consisting of a non empty set S together with an associative binary relation. Their formal study began in the early 20th century. Semigroups importance appears in many mathematical disciplines such as coding and language theory, automata theory, combinatorics and mathematical analysis. Γ-semigroups, as a generalization of semigroups are defined by Sen and Saha in 1986. They have attracted many other mathematicians, who have generalized a lot of classical results from the theory of semigroups. Let us mention here Chattopadhyay, Chinram,Tinpun,Sattayaporn etc.

2 Preliminaries

Let S and Γ be two nonempty sets. S is called a Γ-semigroup ([2]) if there exists a mapping $\cdot : S \times \Gamma \times S \to S$ written as $(x, \gamma, y) \mapsto x\gamma y$ satisfying $(x, \gamma, y)\, \beta z = x\gamma(y\beta z)$ for all $x, y, z \in S$ and $\gamma, \beta \in \Gamma$. In this case by $(S, \Gamma, \cdot)$ we mean S is a Γ- semigroup. For a Γ-semigroup S and a fixed element $\gamma \in \Gamma$ we define on S the binary operation $\circ$ by putting $x \circ y = x\gamma y$ for all $x, y \in S$. The pair $(S, \circ)$ such defined is denoted by S_γ. It is a semigroup. Moreover, if it is a group for some $\gamma \in \Gamma$ then it is a group for every $\gamma \in \Gamma$. In this case we say that S is a Γ-group.

We denote by $\boldsymbol{\Gamma - Sgrp}$ the category of Γ-semigroups which has the Γ-semigroups as objects and the homomorphisms of Γ-semigroups as arrows.

Let S be a Γ-semigroup. A nonempty subset T of S is said to be a Γ-subsemigroup of S if $a\gamma b \in T$, for all $a, b \in T$ and $\gamma \in \Gamma$. We denote this by $T \leq S$.

Let S be a Γ-semigroup and $X \subseteq S, X \neq \emptyset$. We denote by $< X >_S = \bigcap\{A | X \subseteq A, A \leq S\}$. Then, as can be easily verified $< X >_S$ is a Γ-subsemigroup and it is called the Γ-subsemigroup generated by X.

Theorem 2.1. Let $X \neq \emptyset, X \subseteq S$ for a Γ-semigroup S. Then

$$< X >_S = \bigcup_{n=1}^{\infty} X^n = \{x_1\, \alpha_1 x_2 \dots x_{n-1}\alpha_{n-1}x_n | n \geq 1, x_i \in X, \alpha_i \in \Gamma\}$$

Proof: Write $A = \bigcup_{n=1}^{\infty} X^n$.It is easy to see that $A \leq S$.Also, $X^n \subseteq < X >_S$ for all $n \geq 1$, since $< X >_S \leq S$ and hence the claim follows.

Lemma 2.2. Let $\alpha: S \to P$ be a homomorphism of Γ-semigroups. If $X \subseteq S$ then

$\alpha(< X >_S) = < \alpha(X) >_P$.

Proof: If $x \in < X >_S$ then by Theorem 2.1. $x = x_1\alpha_1 x_2 \dots x_{n-1}\alpha_{n-1}x_n$ for some $x_i \in X, \alpha_i \in \Gamma$.Since α is a homomorphism we have

$$\alpha(x) = \alpha(x_1)\alpha_1\alpha(x_{2)} \dots \alpha(x_{n-1)}\alpha_{n-1}\alpha(x_n) \in < \alpha(X) >_P$$

And so $\alpha(< X >_S) \subseteq < \alpha(X) >_P$. On the other hand if $y \in) \subseteq < \alpha(X) >_P$ then again by Theorem 2.1. $y = \alpha(x_1)\alpha_1\alpha(x_{2)} \dots \alpha(x_{n-1})\alpha_{n-1}\alpha(x_n)$ for some $\alpha(x_i) \in \alpha(X)(x_i \in X)$. The claim follows now since α is a homomorphism: $y = \alpha(x_1\alpha_1 x_2 \dots x_{n-1}\alpha_{n-1}x_n)$ where $x_1\alpha_1 x_2 \dots x_{n-1}\alpha_{n-1}x_n \in < X >_S$.

Lemma 2.3. If $\alpha: S \to P$ is an isomorphism of Γ-semigroups then also $\alpha^{-1}: P \to S$ is an isomorphism of Γ-semigroups.

Proof: First of all, α^{-1} exists,because α is a bijection. Furthermore, $\alpha\alpha^{-1} = \iota$, and thus, because α is a homomorphism we have

$$\alpha\left(\alpha^{-1}(x)\gamma\alpha^{-1}(y)\right) = \alpha\left(\alpha^{-1}(x)\right)\gamma\alpha\left(\alpha^{-1}(y)\right) = x\gamma y$$

And so $\alpha^{-1}(x)\gamma\alpha^{-1}(y) = \alpha^{-1}(x\gamma y)$, as desired.

Definition 2.4. An element a of a Γ-semigroup S is said to be cancellative provided it is both left and right α-cancellative.

Definition 2.5. An element a of a Γ-semigroup S is said to be left-Γ-cancellative provided a is left-α-cancellative for all $\alpha \in \Gamma$.

Definition 2.6. An element a of a Γ-semigroup S is said to be right-Γ-cancellative provided a is right-α-cancellative for all $\alpha \in \Gamma$.

Definition 2.7. An element a of a Γ-semigroup S is said to be Γ-cancellative provided it is both left and right Γ-cancellative.

Definition 2.8. A Γ-semigroup S is said to be cancellative provided every $a \in S$ is Γ-cancellative.

Definition 2.9.([3]). Given a Γ-semigroup S we define its universal semigroup Σ as the quotient of the free semigroup F on the set $S \cup \Gamma$ by the congruence generated from the relations $(\gamma_1, \gamma_2) \sim \gamma_1, (x, \gamma, y) \sim x\gamma y, (x, y) \sim x\gamma_0 y$

for all $(\gamma_1, \gamma_2, \gamma \in \Gamma$,all $x, y \in S$ and with $\gamma_0 \in \Gamma$ fixed element.

Lemma 2.10.([3],Lemma 1.1) Every element of Σ can be represented by an irreducible word which has the form $\gamma x \gamma'$,γx,$x\gamma$,γ or x where $x \in S$ and $\gamma, \gamma' \in \Gamma$.

Two sets X and Y have the same cardinality, and this is denoted $|X| = |Y|$, if there exists a bijection, that is, an injective and surjective function, from X to Y, $\varphi: X \to Y$. In this case the function $\varphi^{-1}: Y \to X$ is a bijection, too. So, there is a 1-to-1 correspondence between the elements of X and Y and if X is finite, then $|X| = |Y|$ if and only if X and Y have the exactly the same number of elements,

Let A be a set of symbols, called an alphabet. Its elements are letters and any finite sequence of letters is a word over A. We denote by A^* the set of all words over A. It is a semigroup when the product is defined as the concatenation of words. It is a free semigroup over A, as well.

Proposition 2.11.([7],Theorem 3.4.) A semigroup S is free if and only if $S \cong A^*$, for some alphabet A.

Corollary 2.12. If S is freely generated by a set X, then $S \cong A^*$ where $|A| = |X|$.

Corollary 2.13. If S and R are free semigroups generated by X and Y respectively such that $|X| = |Y|$ then $S \cong R$.

3 Equivalences

As we know, a relation ρ on a set X is: reflexive if and only if $1_X \subseteq \rho$, antisymmetric if and only if $\rho \cap \rho^{-1} = 1_X$, and transitive if and only if $\rho \circ \rho \subseteq \rho$.We define an equivalence ρ on a set X as a relation that is reflexive, transitive and symmetric i.e. such that

$$(\forall x, y \in X)(x, y) \in \rho \Longrightarrow (y, x) \in \rho.$$

We can express this property as $\rho \subseteq \rho^{-1}$. If we denote by $\mathcal{B}_X$ the set of all binary relations on X and define on $\mathcal{B}_X$ an operation $\circ$ by the rule that, for all $\rho, \sigma \in \mathcal{B}_X$,

$$\rho \circ \sigma = \{(x, y) \in X \times X | (\exists z \in X)(x, z) \in \rho \; and \; (z, y) \in \sigma\} \qquad (3.1)$$

then it is easily verified that for all $\rho, \sigma, \tau, \rho_1, \rho_2, \dots, \rho_n \in \mathcal{B}_X$ the following relations hold:

$$\rho \subseteq \sigma \Longrightarrow \rho \circ \tau \subseteq \sigma \circ \tau, \tau \circ \rho \subseteq \tau \circ \sigma \qquad (3.2)$$

$$(\rho \circ \sigma) \circ \tau = \rho \circ (\sigma \circ \tau) \qquad (3.3)$$

$$(\rho^{-1})^{-1} = \rho \qquad (3.4)$$

$$(\rho_1 \circ \rho_2 \circ \dots \circ \rho_n)^{-1} = \rho_1^{-1} \circ \dots \circ \rho_n^{-1} \qquad (3.5)$$

$$\rho \subseteq \sigma \Longrightarrow \rho^{-1} \subseteq \sigma^{-1} \qquad (3.6)$$

Here by ρ^{-1} we denote the converse of ρ for each $\rho \in \mathcal{B}_X$, i.e

$$\rho^{-1} = \{(x, y) \in X \times X | (y, x) \in \rho\}. \qquad (3.7)$$

If ρ is an equivalence on X then the set of ρ-classes, whose elements are the subsets $x\rho$, is called the quotient set of X by ρ and is denoted by X/ρ.The map $\rho^\natural: X \to X/\rho$ defined by

$$x\rho^\natural = x\rho, x \in X \qquad (3.8)$$

is called the natural map.

Proposition 3.1.([1],Prop.1.4.7) If $\varphi: X \to X$ is a map, then $\varphi \circ \varphi^{-1}$ is an equivalence.

We call this equivalence the kernel of φ and write $\varphi \circ \varphi^{-1} = ker\varphi$.

Let R be a relation on X. We denote by R^e the minimum equivalence on X containing R. The family of equivalences containing R is non-empty since $X \times X$ is one such. Then the intersection of all equivalences containing R is an equivalence and it is just the equivalence generated by R that is R^e. Its properties are given by J.M.Howie ([1]).

4 Congruences on Γ-semigroups

In this section we give some known results about congruences on Γ-semigroups.

Definition 4.1.([4]) An equivalence relation ρ on S is called congruence if $x\rho y$ implies that $(x\gamma z)\rho(y\gamma z)$ and $(z\gamma x)\rho(z\gamma y)$ for all $x, y, z \in S$ and $\gamma \in \Gamma$, where by $x\rho y$ we mean $(x, y) \in \rho$.

Let ρ be a congruence relation on (S, Γ). By S/ρ we mean the set of all equivalence classes of the elements of S with respect to ρ that is $S/\rho = \{\rho(x)/x \in S\}$.

Theorem 4.2.([5]) Let ρ be a congruence relation on (S, Γ). Then S/ρ is a Γ- semigroup.

Proof: Let S be a Γ- semigroup and ρ a congruence on S. For $a\rho, b\rho \in S/\rho$ and $\gamma \in \Gamma$, let $(a\rho)\gamma(b\rho) = (a\gamma b)\rho$. This is well-defined because for $a, a', b, b' \in S$ and $\gamma \in \Gamma$ we have:

$a\rho = a'\rho$ and $b\rho = b'\rho \Longrightarrow (a, a'), (b, b') \in \rho \Longrightarrow (a\gamma b, a'\gamma b), (a'\gamma b, a'\gamma b') \in \rho \Longrightarrow (a\gamma b, a'\gamma b') \in \rho \Longrightarrow (a\gamma b)\rho = (a'\gamma b')\rho$.

Now, let $a, b, c \in S$ and $\gamma, \mu \in \Gamma$. Then we have

$(a\rho\gamma b\rho)\mu c\rho = \big((a\gamma b)\rho\big)\mu c\rho = \big(a\gamma(b\mu c)\big)\rho = a\rho\gamma(b\mu c)\rho = a\rho\gamma(b\rho\mu c\rho)$.

This proves the theorem.

Theorem 4.3. ([6]) Let $(\varphi, g): (S_1, \Gamma_1) \to (S_2, \Gamma_2)$ be a homomorphism. Define the relation $\rho_{(\varphi,g)}$ on (S_1, Γ_1) as follows:

$x\rho_{(\varphi,g)}y \Leftrightarrow \varphi(x) = \varphi(y)$. Then $\rho_{(\varphi,g)}$ is a congruence on (S_1, Γ_1).

Proof: Clearly, $\rho_{(\varphi,g)}$ is an equivalence relation. Suppose that $x\rho_{(\varphi,g)}y$. We have $\varphi(x) = \varphi(y) \Rightarrow \varphi(x)g(\gamma)\varphi(z) = \varphi(y)g(\gamma)\varphi(z) \Rightarrow \varphi(x\gamma z) = \varphi(y\gamma z)$ for all $z \in S_1$ and $\gamma \in \Gamma_1$. Thus,$(x\gamma z)\rho_{(\varphi,g)}(y\gamma z)$. In a similar way,we show that $(z\gamma x)\rho_{(\varphi,g)}(z\gamma y)$. Therefore, $\rho_{(\varphi,g)}$ is a congruence relation on (S_1, Γ_1).

Theorem 4.4. ([5],Theorem 2.1.) Let S and T be Γ- semigroups under same Γ and $\phi: S \to T$ be a Γ-homomorphism.Then there is a a Γ-homomorphism $\varphi: S/ker\phi \to T$ such that $im\phi = im\varphi$ and the diagram

$$S \overset{\phi}{\rightarrow} T$$
$$(ker\phi)^{\flat} \downarrow \nearrow \varphi$$
$$S/ker\phi$$

commutes (i.e. $\varphi \circ (ker\phi)^{\flat} = \phi$) where $(ker\phi)^{\flat}$ is the natural mapping from S onto $S/ker\phi$

defined by $(ker\phi)^{\flat}(x) = xker\phi$ for all $x \in S$.

Corollary 4.4.1. Let Let S and T be Γ- semigroups under same Γ and $\phi: S \rightarrow T$ be a Γ-homomorphism.Then $S/ker\phi \cong im\phi$.

Theorem 4.5.([6] Isomorphism theorem): If $\varphi: S_1 \rightarrow S_2$ is a homomorphism of Γ-semigroups with the same Γ then there exists a unique isomorphism $\psi: S_1/\rho \rightarrow S_2$ such that the following diagram commutes:

$$S_1 \overset{\varphi}{\rightarrow} S_2$$
$$\Pi_{S_1} \downarrow \nearrow \psi$$
$$S_1/\rho_{\varphi}$$

where $\Pi_{S_1}: S_1 \rightarrow S_1/\rho_{\varphi}$ is defined by $\Pi_{S_1}(x) = \rho_{\varphi}(x)$ for all $x \in S_1$.

Let ρ and σ be congruences on a Γ-semigroup S with $\rho \subseteq \sigma$. Define the relation σ/ρ on S/ρ by

$$\sigma/\rho = \{(x\rho, y\rho) \in S/\rho \times S/\rho | (x, y) \in \sigma\}$$

To show that σ/ρ is well-defined, let $x\rho, a\rho, y\rho, b\rho \in S/\rho$ such that $x\rho = a\rho$ and $y\rho = b\rho$.Thus $(x, a), (y, b) \in \rho$. Since $\rho \subseteq \sigma$, $(x, a), (y, b) \in \sigma$. It follows that $(x, y) \in \sigma \Leftrightarrow (a, b) \in \sigma$.

Theorem 4.6.([5]) Let ρ and σ be congruences on a Γ-semigroup S with $\rho \subseteq \sigma$ and

$\sigma/\rho = \{(x\rho, y\rho) \in S/\rho \times S/\rho | (x, y) \in \sigma\}$.

Then (i) σ/ρ is a congruence on S/ρ and (ii) $(S/\rho)/(\sigma/\rho) \cong S/\sigma$.

5 Construction of Free Γ-semigroups

Let X and Γ be two nonempty sets. A sequence of elements $x_1\alpha_1x_2\alpha_2 \dots x_{n-1}\alpha_{n-1}x_n$ where $x_1, x_2, \dots, x_n \in X$ and $\alpha_1, \alpha_2, \dots, \alpha_{n-1} \in \Gamma$ is called a word over the alphabet X relative to Γ.The set S of all words with the operation defined from $S \times \Gamma \times S$ to S as $(x_1\alpha_1x_2\alpha_2 \dots x_{n-1}\alpha_{n-1}x_n)\gamma(y_1\beta_1y_2\beta_2 \dots y_{m-1}\beta_{m-1}y_m) =$ $x_1\alpha_1x_2\alpha_2 \dots x_{n-1}\alpha_{n-1}x_n\gamma y_1\beta_1y_2\beta_2 \dots y_{m-1}\beta_{m-1}y_m$

is a Γ-semigroup. This Γ-semigroup is called free Γ-semigroup over the alphabet X relative to Γ and we denote it by $X^*\Gamma$. For clarity,we shall often write $u \equiv v$, if the words u and v are the same (letter by letter). The empty word is the word which has no letters. Hence,

$$X^*\Gamma = \{x_1\alpha_1x_2\alpha_2 \dots x_{n-1}\alpha_{n-1}x_n | \alpha_1, \alpha_2, \dots, \alpha_{n-1} \in \Gamma, x_1, x_2, \dots, x_n \in X\}$$

Closely related to the forgetful functor $\mathcal{U}: \boldsymbol{\Gamma - Sgrp} \to \boldsymbol{Set}$ such that $(S, \Gamma, \cdot) \mapsto S$ is the functor F: $\boldsymbol{Set} \to \boldsymbol{\Gamma - Sgrp}$ defined as follows: $X \mapsto (X^*\Gamma, \Gamma, \cdot)$.

For a function $f: X \to Y$ define $F(f): (X^*\Gamma, \Gamma, \cdot) \to (Y^*\Gamma, \Gamma, \cdot)$ such that

$$F(f)(x_1, x_2, \dots, x_n) = f(x_1)\gamma_1 f(x_2)\gamma_2 \dots f(x_{n-1})\gamma_{n-1} f(x_n)$$

where $x_i = a_1^i \alpha_1^i \dots a_{m-1}^i \alpha_{m-1}^i a_m^i$,*i=1,2,…,n*.

F as so defined is a functor.

Now suppose that $f: X \to \mathcal{U}(Y, \Gamma, \cdot)$ is any function from a set X to (the underlying set) of a Γ-semigroup Y. Then we can define a Γ-semigroup homomorphism $f^*: X^*\Gamma \to Y$ by

$$f^*(x_1, x_2, \dots, x_n) = f(x_1)\gamma_1 f(x_2)\gamma_2 \dots f(x_{n-1})\gamma_{n-1} f(x_n)$$

where $x_i = a_1^i \alpha_1^i \dots a_{m-1}^i \alpha_{m-1}^i a_m^i$,*i=1,2,…,n*.

Clearly, f^* is the unique Γ-semigroup homomorphism extending *f*, i.e.if $h: X^*\Gamma \to Y$ is a Γ-semigroup homomorphism such that $h(x) = f(x)$ for every $x \in X$ then $h = f^*$. Indeed,let $\iota: X \hookrightarrow X^*\Gamma$ be the embedding map and f as above. Define f^* as above, as well. Then $\iota f^* = f$. Now,let $h: X^*\Gamma \to Y$ be an arbitrary homomorphism with $\iota h = f$. For any $x_1, x_2, \dots, x_n \in X^*\Gamma$

$h(x_1, x_2, \dots, x_n) = f(x_1)\gamma_1 f(x_2)\gamma_2 \dots f(x_{n-1})\gamma_{n-1} f(x_n) = f^*(x_1, x_2, \dots, x_n)$ which implies that $h = f^*$.

This constitutes the socalled Universal Mapping Property for the free Γ-semigroup $X^*\Gamma$ generated by X. Another way of stating this result is that we have a function $\boldsymbol{Set}(X, \mathcal{U}(Y, \Gamma, \cdot)) \to \boldsymbol{\Gamma - Sgrp}((X^*\Gamma, \Gamma, \cdot), Y)$ which is a bijection. It's in fact an isomorphism and $\mathcal{U}$ and F are a pair of adjoint functors.

Proposition 5.1. Let X be an alphabet and F let be a Γ-semigroup. Then F is a free Γ-semigroup on X relative to Γ if and only if $F \cong X^*\Gamma$.

Proof: Suppose $F \cong X^*\Gamma$ .To show that F is a free Γ-semigroup on X, it is sufficient to show that $X^*\Gamma$ is a free semigroup on X. Let $\iota: X \to X^*\Gamma$ be the embedding map. So let S be a Γ-semigroup and $\varphi: X \to S$ be a map. Define $\varphi^*: X^*\Gamma \to S$ by

$$\varphi^*(x_1, x_2, \dots, x_n) = \varphi(x_1)\gamma_1\varphi(x_2)\gamma_2 \dots \varphi(x_{n-1})\gamma_{n-1}\varphi(x_n)$$

It is easy to see that φ^* is a homomorphism and that $\iota\varphi^* = \varphi$. We now have to prove that φ^* is unique. So let $\psi: X^*\Gamma \to S$ be an arbitrary homomorphism with $\iota\psi = \varphi$. Then for any $x_1, \dots, x_n \in X^*\Gamma$, we have

$$\psi(x_1, x_2, \dots, x_n) = \psi(x_1)\gamma_1\psi(x_2)\gamma_2 \dots \psi(x_{n-1})\gamma_{n-1}\psi(x_n)$$

$$= \varphi^*(x_1)\gamma_1\varphi^*(x_2)\gamma_2 \dots \varphi^*(x_{n-1})\gamma_{n-1}\varphi^*(x_n)$$

$$= \varphi^*(x_1, x_2, \dots, x_n)$$

These equalities hold because ψ is a homomorphism, $\iota\psi = \varphi = \iota\varphi^*$ and φ^* is a homomorphism,too. Hence,$\psi = \varphi^*$. Thus, φ^* is the unique homomorphism from $X^*\Gamma$ to S with $\iota\varphi^* = \psi$, and so $X^*\Gamma$ is free on X.

Let, now, F be a free Γ-semigroup on X relative to Γ. Let $\iota_1: X \hookrightarrow X^*\Gamma$ and $\iota_2: X \hookrightarrow F$ be the embedding maps. Putting $\varphi = \iota_2$ and $S = F$ in the definition of freeness for F on X we see that there is a homomorphism $\iota_2^*: X^*\Gamma \to F$ with $\iota_1\iota_2^* = \iota_2$. Similarly, since F is free on X there is a homomorphism $\iota_1^*: F \to X^*\Gamma$ with $\iota_2\iota_1^* = \iota_1$. Therefore $\iota_1 = \iota_1\iota_2^*\iota_1^*$ and $\iota_2 = \iota_2\iota_1^*\iota_2^*$. Hence,by the uniqueness requirement in the definition of freeness, we have $\iota_2^*\iota_1^* = id_A$ and $\iota_1^*\iota_2^* = id_F$. Thus, ι_1^* and ι_2^* are mutually inverse homomorphisms and so $\cong X^*\Gamma$.

A family $\mathcal{V}$ of Γ-semigroups is called a variety of Γ-semigroups if it contains Γ-subsemigroups, all homomorphic images and all direct products of its elements.

We say that $\mathcal{V}$ is generated by $\mathcal{U} \subseteq \mathcal{V}$ if $\mathcal{V}$ is the smallest variety containing $\mathcal{U}$. This is equivalent to every member of $\mathcal{V}$ being obtainable from algebras in $\mathcal{U}$ via a sequence of taking homomorphic images, subalgebras and direct products (H,S and P).

Theorem 5.2. A variety $\mathcal{V}$ is generated by $\mathcal{U} \subseteq \mathcal{V}$ if and only if every $A \in \mathcal{V}$ is in $HSP(\mathcal{U})$ i.e. there exist $\mathcal{U}_\alpha \in \mathcal{U}$ and $T \in \mathcal{V}$, which is a subalgebra of $\prod_{\alpha\in\Lambda} \mathcal{U}_\alpha$ (where Λ is an indexing set) and an onto morphism $\varphi: T \to A$.(see [8]).

The following proposition also holds:

Proposition 5.3. Let $\mathcal{V}$ be a variety and let $\mathcal{U}$ consists of the free objects of $\mathcal{V}$. Then $\mathcal{V}$ is generated by $\mathcal{U}$. (see [8], Proposition 1.4.4.).

The following theorem is a generalization of Theorem 3.3. in [7]. Its proof is the same as that of Theorem 3.3. in [7], but for the reader's convienence we will give its proof here.

Theorem 5.4. For each Γ-semigroup S there exists an alphabet Y and an epimorphism $\psi: Y^*\Gamma \twoheadrightarrow S$.

Proof: Let X be any generating set of S; we may even choose as X the set S itself. Let Y be an alphabet such that $|Y| = |X|$. Let $\psi_0: Y \to X$ be a bijection. By definition of the free Γ-semigroup, the bijection ψ_0 has a homomorphic extension $\psi: Y^*\Gamma \to S$. This extension is surjective, since $< \psi(X) >_S = \psi(< X >_S) = \psi(S)$, (because X generates S).

Corollary 5.4.1. Every Γ-semigroup is a quotient of a free semigroup.Indeed

$S \cong Y^*\Gamma/\ker(\psi)$ for a suitable epimorphism ψ.

Let $X \subseteq S$, where S is a Γ-semigroup. We say that $x = x_1\alpha_1x_2\alpha_2 \ldots x_{n-1}\alpha_{n-1}x_n$ is a factorization of x over X relative to Γ. Usually, this factorization is not unique,but…

Theorem 5.5. A Γ-semigroup S is freely generated by Y if and only if every $x \in S$ has a unique factorization over Y relative to Γ.

Proof: We observe, first, that the claim holds for the word semigroup $X^*\Gamma$, for which X is the only minimal generating set. Let X be an alphabet such that $|X| = |Y|$ and let $g_0: Y \to X$ be a bijection. Suppose that Y generates S freely and that there is an $x \in S$,for which

$$x = x_1\alpha_1x_2\alpha_2 \ldots x_{n-1}\alpha_{n-1}x_n = y_1\beta_1y_2\beta_2 \ldots y_{m-1}\beta_{m-1}y_m\, , (x_i, y_j) \in X, (\alpha_i, \beta_j) \in \Gamma$$

For the homomorphic extension g of g_0 we have

$$g(x) = g_0(x_1)\alpha_1 g_0(x_2)\alpha_2 \dots g_0(x_{n-1})\alpha_{n-1} g_0(x_n)$$
$$= g_0(y_1)\beta_1 g_0(y_2)\beta_2 \dots g_0(y_{m-1})\beta_{m-1} g_0(y_m)$$

in $X^*\Gamma$. Since $X^*\Gamma$ satisfies the condition of the theorem and $g_0(x_i), g_0(y_i)$ are letters for each i,we must have $g_0(x_i) = g_0(y_i)$ for all $i = 1,2, \dots, n$ (and $m = n$). Moreover, g_0 is injective,and so $x_i = y_i$. Hence $\alpha_i = \beta_i$ for all $i = 1,2, \dots, n$. Thus the claim holds for S, also. Suppose , now that S satisfies the uniqueness condition. Denote by $h_0 = g_0^{-1}$ and let $h: X^*\Gamma \to S$ be the homomorphic extension of h_0. But, h is surjective, because Y generates S. It is also injective, because if $h(u) = h(v)$ for some words $u \neq v \in X^*\Gamma$,then $h(u)$ would have two different factorizations over Y. Hence h is an isomorphism,and the claim is proved.

6 Some properties of free Γ-semigroups

Proposition 6.1.The universal semigroup Σ of a free Γ-semigroup is not a free semigroup but there is a subset $S = \{x_1\alpha_1 x_2\alpha_2 \dots x_{n-1}\alpha_{n-1}x_n : x_i \in X, \alpha_i \in \Gamma, i = 1,2, \dots, n\}$ of Σ such that for the pair $(S,\circ)$ where " $\circ$ " is defined as follows: $w_1 \circ w_2 = w_1\gamma_0 w_2$ (we shall denote it by S_{γ_0}) is free on Y where $Y = \{ x_1\alpha_1 x_2\alpha_2 \dots x_{n-1}\alpha_{n-1}x_n : x_i \in X, \alpha_i \in \Gamma, \alpha_i \neq \gamma_0, \forall i = 1,2, \dots, n\}$.

Proof: The universal semigroup Σ of a free Γ-semigroup is not a free semigroup because, by Lemma 2.10., it follows that there exist relations between the words such that, for example, $\alpha = \alpha\beta$. From the Proposition 5.1., it follows that to show that S_{γ_0} is free we have to show that $S_{\gamma_0} \cong Y^*\Gamma$, where $Y^*\Gamma$ is free. Let us show first that $Y^*\Gamma$ is free where from the construction $Y \subset X$. We know that $X^*\Gamma$ is free on X. That is the UMP is satisfied i.e. the following diagram commutes.

$$\begin{array}{c} X \hookrightarrow X^*\Gamma \\ \varphi \searrow \downarrow \varphi^* \\ T \end{array}$$

Now, let we see the corresponding diagram

$$Y \hookrightarrow Y^*\Gamma$$
$$\varphi|_Y \searrow \downarrow \varphi^*|_{Y^*\Gamma}$$
$$T$$

It is obvious that this diagram commutes as well. This means that $Y^*\Gamma$ is a free semigroup on Y. But, it is clear that $S_{\gamma_0} \cong Y^*\Gamma$ (they have the same base). So, by the Proposition 5.1., it follows that S_{γ_0} is free on Y.

Let us denote with $f^*: (S_1 \cup \Gamma)^*/\rho_1 \to (S_2 \cup \Gamma)^*/\rho_2$ such that $f^*(\rho_1(x)) = \rho_2(f(x))$ where $f: S_1 \to S_2$ is a homomorphism of Γ-semigroups. We observe that if $x = y \Longrightarrow f(x) = f(y)$. Then we will have $\rho_2(f(x)) = \rho_2(f(y))$ which implies that $f^*(\rho_1(x)) = f^*(\rho_1(y))$. Therefore, f^* is well defined. Next, we prove that f^* is a homomorphism. But, by the definition of f^* and the fact that f is a homomorphism we will have:

$$f^*(\rho_1(x\gamma y)) = \rho_2(f(x\gamma y)) = \rho_2(f(x)\gamma f(y)) = \rho_2(f(x))\gamma\rho_2(f(y)) =$$
$$f^*(\rho_1(x))\gamma f^*(\rho_1(y))$$

Thus, f^* is a homomorphism.

Now, we construct a functor F between a Γ-semigroup S and its universal semigroup Σ as follows:

$F(S) = \Sigma = (S \cup \Gamma)^*/\rho$ and $F(f) = f^*$ where f is a homomorphism of Γ-semigroups. Let $\psi: S_1 \to S_2$ and $\varphi: S_2 \to S_3$ be homomorphisms of Γ-semigroups. We have $\varphi \circ \psi: S_1 \to S_3$ and we prove that $(\varphi \circ \psi)^* = \varphi^* \circ \psi^*$. But,

$$(\varphi \circ \psi)^*(\rho_1(x)) = \rho_3(\varphi \circ \psi(x)) = \rho_3\left(\varphi(\psi(x))\right) = \varphi^*\left(\rho_2(\psi(x))\right) = \varphi^* \circ \psi^*(\rho_1(x))$$

Thus, $(\varphi \circ \psi)^* = \varphi^* \circ \psi^*$. Therefore, $F(\varphi \circ \psi) = F(\varphi) \circ F(\psi)$. Let $id_S: S \to S$ be the identity homomorphism of the Γ-semigroup S.We have $F(id_S) = id_S^* = id_{(S \cup \Gamma)/\rho}$, because id_S^* and $id_{(S \cup \Gamma)^*/\rho}$ are identity homomorphisms of $(S \cup \Gamma)^*/\rho$. Therefore, F is a covariant functor.

From the Proposition 6.1., it follows that the results of Howie can be implanted on Γ-semigroups through the mechanism of passing from the Γ-semigroup to its universal semigroup associated to Γ. So,we now can formulate and prove these properties of free Γ-semigroups.

Proposition 6.2.The free monoid $MX^*\Gamma$ is cancellative.

Proof: This follows from the fact that two words in the alphabet X represent the same element of $MX^*\Gamma$ if and only if they are identical.

7 Presentations of Γ-semigroups

Let S be a Γ-semigroup.By Theorems 4.5, 5.4. and its Corollary 5.4.1., it follows that

$$S \cong Y^*\Gamma/\ker(\psi)$$

(where $\psi: Y^*\Gamma \twoheadrightarrow S$ is an epimorphism and $Y^*\Gamma$ a suitable word Γ-semigroup), since now $\psi(Y^*\Gamma) = S$. We say that ψ is a homomorphic presentation of S. The letters in Y are called generators of S, and if $(u, v) \in \ker(\psi)$, (which means that $\psi(u) = \psi(v)$) then $u = v$ is called a relation (or an equality) in S. Define a presentation of S as $S =< Y|R >$ ($Y = \{y_1, \dots, y_n\}$ and $R = \{u_i = v_i | i \in I\}$) if $\ker(\psi)$ is the smallest congruence of $Y^*\Gamma$ that contains the relation $\{(u_i, v_i)| i \in I\}$. In particular,

$\psi(u_i) = \psi(v_i)$ for all $u_i = v_i$ in R. (7.1)

The set R of relations is supposed to be symmetric, that is , $u = v \Longrightarrow v = u$ where $u = v$ is in R. Recall that the words $w \in Y^*\Gamma$ are not elements of S but of the word semigroup $Y^*\Gamma$, which is mapped onto S. We say that a word $w \in Y^*\Gamma$ presents the element $\psi(w)$ of S. The same element can be presented by many different words , but if $\psi(u) = \psi(v)$, then both u and v present the same element of S.

Let $S =< Y|R >$ be a presentation. Then, S satisfies a relation $u = v$ (that is, $\psi(u) = \psi(v)$) if and only if there exists a finite sequence $u = u_1, u_2, \dots, u_{k+1} = v$ of words such that u_{i+1} is obtained from u_i by substituting a factor u_i by v_i for some $u_i = v_i$ in R.

So, we say that a word v is directly derivable from the word u, if

$u \equiv w_1u'w_2$ and $v \equiv w_1v'w_2$ for some $u' = v'$ in R. (7.2)

(In order to avoid confusion we use the symbol '$\equiv$' for the equality of two words in $Y^*\Gamma$). It is clear that if v is derivable from u, then u is derivable from v (R is supposed to be symmetric), and, in the notation of (7.2),

$$\psi(u) = \psi(w_1u'w_2) = \psi(w_1)\psi(u')\psi(w_2) = \psi(w_1)\psi(v')\psi(w_2) = \psi(w_1v'w_2) = \psi(v)$$

Thus, $u = v$ is a relation in S.

The word v is derivable from u, if there exists a finite sequence $u \equiv u_1, u_2, \dots, u_k \equiv v$ such that for all $j = 1,2,\dots,k-1$, u_{j+1} is directly derivable from u_j. If v is derivable from u, then $\psi(u) = \psi(v)$, too, because $\psi(u) = \psi(u_1) = \cdots = \psi(u_k) = \psi(v)$. So, $u = v$ is a relation in S. This can be written as

$$u \equiv u_1 = \cdots = u_k \equiv v$$

We denote by R^c the smallest congruence containing R.

Theorem 7.1. Let $S =< Y|R >$ be a presentation (with R symmetric). Then

$$R^c = \{(u,v)|u = v \; or \; v \; is \; derivable \; from \; u\}$$

Hence $u = v$ if and only if v is derivable from u.

Proof: Define the relation ρ by

$u\rho v \Leftrightarrow u = v$ or v is derivable from u.

It can be easily seen that the relation ρ so defined is a congruence of Γ-semigroups. First, it is clear that $\iota_S \subseteq \rho$, and so ρ is reflexive. Since R is symmetric, so is ρ. The transitivity of ρ is easily verified. This shows that ρ is an equivalence relation. In the case $u\rho v \Leftrightarrow u = v$ we can easily verify that $(u\gamma z)\rho(v\gamma z)$ and $(z\gamma u)\rho(z\gamma v)$ also hold. So, ρ is a congruence of Γ-semigroups in this case. Now, if $w \in Y^*\Gamma$ and v is derivable from u, then it is clear that wv is derivable from wu and vw is derivable from uw, too. Thus, ρ is a congruence of Γ-semigroups. Let σ be any congruence such that $R \subseteq \sigma$. Suppose that v is directly derivable from u. This means that $u \equiv w_1 u' w_2$ and $v \equiv w_1 v' w_2$ with $u' = v'$ in R. Since $R \subseteq \sigma$, $(u', v') \in \sigma$ as well and since σ is a congruence, also $(w_1 u' w_2, w_1 v' w_2) \in \sigma$, that is $u\sigma v$. Now, by transitivity of ρ and σ, it follows that $\rho \subseteq \sigma$. Thus, ρ is the smallest congruence that contains R, that is, $\rho = R^c$.

Theorem 7.2. Let Y be an alphabet and $R \subseteq Y^*\Gamma \times Y^*\Gamma$ a symmetric relation. The Γ-semigroup $S = Y^*\Gamma / R^c$, where R^c is the smallest congruence containing R, has the presentation

$$S =< Y|u = v \; for \; all \; (u,v) \in R >$$

Moreover, all Γ-semigroups having a common presentation are isomorphic.

Proof: It follows immediately from the above.

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