

**Multiobjective Variational Problems Involving
Higher Order Derivatives**

B.C. Dash¹, D.K. Dalai²

- 1. Department of Mathematics, Choudwar College, Choudwar,
Cuttack,
Odisha, India**

Email id. bikashchandra1959@gmail.com, Mob.-(0)7873464222

- 2. Principal, Govt. Science College, Ayeba, Kendrapara, Odisha, India**

Email id. drdalai@yahoo.com, Mob.-(0) 8280129871

Abstract :

In this paper two types of duals are considered for a class of variational problems involving 3rd order derivatives. The duality results are derived without any use of optimality conditions. One set of results is based on Mond weir type dual that has the same objective functional as the primal problem but different constraints. The second set of results is based on a dual of an auxiliary primal with single objective function. Under various convexity and generalized convexity assumptions, duality relationships between primal and its various duals are established. Problems with natural boundary values are considered and the analogs of our results in nonlinear programming are also indicated.

Our results are generalizations of those presented by I. Husain, B.Ahmad and Z.Jabeen.

Keywords : Variational Problems, Multiobjective duality, Concavity, Generalized convexity.

1. INTRODUCTION

Calculus of variations offers a powerful technique for the solution of various important problems appearing in dynamics of rigid bodies, optimization of orbits, theory of vibrations and many areas of science and engineering. The subject of calculus of variation primarily concerns with finding optimal value of a definite integral involving a certain function subject to fixed point boundary conditions. Mond and Hanson [1] were the first to present the problem of calculus of variation as a mathematical programming problem in infinite dimensional space. Since that time many researches contributed to this subject extensively.

In this paper, we consider a vector valued function for the primal problem and its minimality in the Pareto sense. Both equality and inequality constraints are considered in the formulations. In establishing duality results we consider two types of dual problems to the primal problem. The first one has vector valued objective whereas the second set of results based on the duality relations between an auxiliary problems and its associated dual as defined in Mond and Hanson [1]. Duality theorems, unlike in case of classical mathematical programming, are not based on optimality criteria but on certain types of convexity and generalized convexity requirements. Finally multiobjective variational problems with natural boundary values rather than fixed end points are mentioned and the analogs of our results in nonlinear programming are pointed out.

We further generalized the work of I. Husain, B.Ahmad and Z.Jaben [9] taking into account of 3rd ordered instead of 2nd ordered derivatives.

2. PRE-REQUISITES

In the following variational Problem (VP), by minimally we mean Pareto minimally. Now consider the following multiobjective variational problem involving 3rd order derivatives.

$$\text{(VP) Minimize : } \left(\int_1 f^1(t, x, x', x'', x''') dt, \dots, \int_1 f^p(t, x, x', x'', x''') dt, \right)$$

subject to

$$x(a) = x(b) = 0 \quad (1)$$

$$x'(a) = x'(b) = 0 \quad (2)$$

$$g(t, x, x', x'', x''') \leq 0, t \in I \quad (3)$$

$$h(t, x, x', x'', x''') = 0, t \in I \quad (4)$$

where (i) for $I = [a, b] \subseteq \mathbb{R}$, $f : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$,

$g : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m$, and $h : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^k$, are continuously differentiable functions, and

(ii) X denotes the space of piecewise smooth functions $x : I \rightarrow \mathbb{R}^n$ having its first, second and third order derivatives x', x'', x''' respectively having norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty + \|D^3x\|_\infty$, where the differentiation operator D is given by

$$\omega = Dx \Leftrightarrow x(t) = \int_a^t \omega(s)ds, \text{ Thus } D = \frac{d}{dt} \text{ except at discontinuities.}$$

We denote the set of feasible solutions of the problem (VP) by K_p i.e.,

$$K_p = \left\{ x \in X \left| \begin{array}{l} x(a) = x(b) = 0, \quad g(t, x', x'', x''') \leq 0, \quad t \in I \\ x'(a) = x'(b) = 0, \quad h(t, x, x', x'', x''') = 0, \quad t \in I \end{array} \right. \right.$$

The following convention for inequalities for vectors in $\mathbb{R}^n$ given in Mangasarian [4] will be used throughout the development of the theory.

If $x, y \in \mathbb{R}^n$, then $x \geq y \Leftrightarrow x_i \geq y_i, i = 1, 2, \dots, n$

$x \geq y \Leftrightarrow x_i \geq y_i \text{ and } x \neq y$

$x > y \Leftrightarrow x_i > y_i, i = 1, 2, 3, \dots, n$

2.1 Definition :

A feasible solution of the problem (VP) i.e., $x \in K_p$ is said to be Pareto minimum if there exists no $y \in K_p$ such that

$$\left(\int_I f^i(t, x, x', x'', x''')dt, \dots, \int_I f^p(t, x, x', x'', x''')dt \right) \\ \leq \left(\int_I f^i(t, y, y', y'', y''')dt, \dots, \int_I f^p(t, y, y', y'', y''')dt \right)$$

Pareto maximality can be defined in the same way except that the inequality in the above definition is reversed.

In the subsequent analysis the following result plays a significant role.

2.1.1 PROPOSITION : Suppose $\lambda > 0, \lambda \in \mathbb{R}^p$ such that it satisfies the primal condition (P_λ) . Let $y(t) \in K_p$ be an optimal solution of the problem.

$$(P_1) : \text{Min} \int_1^T \lambda^T f(t, x, x', x'', x''') dt$$

Then $y(t)$ is an optimal solution of (MP) in the Pareto sense.

Proof : Assume $y(t)$ is not a Pareto optimal of (MP). Then there exists an $x(t) \in K_p$ such that

$$\int_1^T f^i(t, x, x', x'', x''') dt \leq \int_1^T f^i(t, y, y', y'', y''') dt, i = 1, 2, \dots, p.$$

$$\int_1^T f^i(t, x, x', x'', x''') dt < \int_1^T f^j(t, y, y', y'', y''') dt, i \neq j.$$

$$\text{Hence } \int_1^T \lambda^T f(t, x, x', x'', x''') dt < \int_1^T \lambda^T f(t, y, y', y'', y''') dt.$$

This contradicts the assumption that $y(t)$ is an optimal solution.

In the following sections some duality results (two types of duals to VP) will be established.

3. MOND-WEIR TYPE MULTIOBJECTIVE DUALITY

Consider the following Mond-Weir [2] dual to (VP)

(M-WD) : Maximize

$$\left(\int_1^T f^i(t, u, u', u'', u''') dt, \dots, \int_1^T f^p(t, u, u', u'', u''') dt \right)$$

Subject to

$$u(a) = u(b) = 0 \quad (5)$$

$$u'(a) = u'(b) = 0 \quad (6)$$

$$\lambda^T f_u(t, u, u', u'', u''') + s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''')$$

$$- D(\lambda^T f_{u'}(t, u, u', u'', u''') + s(t)^T g_{u'}(t, u, u', u'', u''') + r(t)^T h_{u'}(t, u, u', u'', u'''))$$

$$+ D^2(\lambda^T f_{u''}(t, u, u', u'', u''') + s(t)^T g_{u''}(t, u, u', u'', u''') + r(t)^T h_{u''}(t, u, u', u'', u'''))$$

$$- D^3(\lambda^T f_{u'''}(t, u, u', u'', u''') + s(t)^T g_{u'''}(t, u, u', u'', u''') + r(t)^T h_{u'''}(t, u, u', u'', u''')) = 0 \quad (7)$$

$$\int_I (r(t)^T g(t, u, u', u'', u''') + s(t)^T h(t, u, u', u'', u''')) \geq 0 \quad (8)$$

$$\lambda > 0, \quad \lambda \in \mathbb{R}^n, \quad s(t) \geq 0, \quad t \in I \quad (9)$$

Let K_p be the set of the feasible solutions of (M-WD).

3.1 THEOREM : Suppose

$$(A_1) : y(t) \in K_p$$

$$(A_2) : (\lambda, u, s(t), r(t)) \in K_p$$

$$(A_3) : \int_I \lambda^T f(t, \dots) dt \text{ is pseudo-convex}$$

$$(A_4) : \int_I \{s(t)^T g(t, \dots) + r(t)^T h(t, \dots)\} dt \text{ is quasiconvex.}$$

$$\text{Then } \int_I \lambda^T f(t, x, x', x'', x''') dt \geq \int_I \lambda^T f(t, u, u', u'', u''') dt$$

Proof : Since $s(t) \geq 0, t \in I, g(t, x, x', x'', x''') \leq 0, t \in I$ and $h(t, x, x', x'', x''') \leq 0, t \in I$, we have

$$\int_I (s(t)^T g(t, x, x', x'', x''') + r(t)^T h(t, x, x', x'', x''')) dt \leq 0 \quad (10)$$

Combining this with (8) we, have

$$\begin{aligned} \int_I (s(t)^T g(t, x, x', x'', x''') + r(t)^T h(t, x, x', x'', x''')) dt \\ \leq \int_I (s(t)^T g(t, u, u', u'', u''') + r(t)^T h(t, u, u', u'', u''')) \end{aligned}$$

By the hypothesis (A_4) , this yields

$$\begin{aligned} 0 \geq \int_I & [(x - u)^T (s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''')) + \\ & (x' - u')^T (s(t)^T g_{u'}(t, u, u', u'', u''') + r(t)^T h_{u'}(t, u, u', u'', u''')) + \\ & (x'' - u'')^T (s(t)^T g_{u''}(t, u, u', u'', u''') + r(t)^T h_{u''}(t, u, u', u'', u'''))] dt \end{aligned}$$

By using integration by parts, we get

$$\begin{aligned}
 & \int_1 (x-u)^T \left(s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''') \right) dt + \\
 & \quad (x-u)^T \left(s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''') \right) \Big|_{t=a}^{t=b} - \\
 & \int_1 (x-u)^T D \left(s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''') \right) dt + \\
 & \quad (x'-u')^T \left(s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''') \right) \Big|_{t=a}^{t=b} - \\
 & \quad (x'-u')^T D \left(s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''') \right) \\
 & = \int_1 (x-u)^T \left[\left(s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''') \right) \right. \\
 & \quad \left. - D \left(s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''') \right) \right] dt \\
 & = (x-u)^T D \left(s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''') \right) + \\
 & \quad \int_1 (x-u)^T D^2 \left(s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''') \right) dt \text{ (by integration by parts)}
 \end{aligned}$$

Using (7) we get

$$\begin{aligned}
 0 & \leq \int_1 (x-u)^T \left\{ \lambda^T f_u(t, u, u', u'', u''') - D \lambda^T f_u(t, u, u', u'', u''') + D^2 \lambda^T f_u(t, u, u', u'', u''') \right\} dt \\
 0 & \leq \int_1 \left\{ (x-u)^T \lambda^T f_u(t, u, u', u'', u''') + (x'-u')^T \lambda^T f_u(t, u, u', u'', u''') \right\} dt - \\
 & \quad (x-u)^T \lambda^T f_u(t, u, u', u'', u''') \Big|_{t=a}^{t=b} - (x'-u')^T D \lambda^T f_u(t, u, u', u'', u''') dt + \\
 & \quad (x'-u')^T \lambda^T f_u(t, u, u', u'', u''') \Big|_{t=a}^{t=b}
 \end{aligned}$$

$$= \int_1 \left\{ (x-u)^T \lambda^T f_u(t, u, u', u'', u''') + (x'-u')^T D \lambda^T f_{u''}(t, u, u', u'', u''') \right\} dt + \\ (x''-u'')^T D \lambda^T f_{u'''}(t, u, u', u'', u''') dt \geq 0$$

Thus using by integration by parts, the boundary conditions, and by using A_3 we get

$$\int_1 \lambda^T f(t, x, x', x'', x''') dt \geq \int_1 \lambda^T f(t, u, u', u'', u''') dt$$

3.1.1 Theorem : Assume

$$(B_1) : y(t) \in K_p$$

$$(B_2) : (\lambda, \bar{u}(t), \bar{s}(t), \bar{r}(t)) \in K_p$$

$$(B_3) : \int_1 (y(t)^T g(t, \dots) + r(t)^T h(t, \dots)) dt \text{ is quasi-convex}$$

$$(B_4) : \int_1 \lambda^T f(t, \dots) dt \text{ is pseudo convex}$$

$$(B_5) : \int_1 \bar{\lambda}^T f(t, y, y', y'', y''') dt = \int_1 \bar{\lambda}^T f(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')$$

Then $y(t)$ is an optimal solution of (VP) and $(\lambda, \bar{u}(t), \bar{s}(t), \bar{r}(t))$ is an optimal solution of the problem (M-WD).

Proof :

Assume that $y(t)$ is not Pareto-optimal of (VP). Then there exist an $x(t) \in K_p$ such that

$$\int_1 f^i(t, x, x', x'', x''') dt \leq \int_1 f^i(t, y, y', y'', y''') dt \text{ for all } i$$

$$\text{and } \int_1 f^j(t, x, x', x'', x''') dt \leq \int_1 f^j(t, y, y', y'', y''') dt \text{ for some } j, 1 \leq j \leq p.$$

Since $\bar{\lambda} > 0$, this implies,

$$\int_1 \bar{\lambda}^T (t, x, x', x'', x''') dt < \int_1 \bar{\lambda}^T (t, y, y', y'', y''') dt$$

By hypothesis (B_5), this inequality implies,

$$\int_1 \bar{\lambda}^T(t, x, x', x'', x''') dt < \int_1 \bar{\lambda}^T(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') dt$$

This contradicts the conclusion of 3.1 Theorem thus establishing the Pareto optimality of $y(t)$ for (VP). Similarly we can show that $(\lambda, \bar{u}(t), \bar{s}(t), \bar{r}(t))$ is Pareto optimal for (M-WD).

3.1.2 Theorem : Assume,

$$(B_1) : y(t) \in K_p$$

$$(B_2) : (\lambda, \bar{u}(t), \bar{s}(t), \bar{r}(t)) \in K_p$$

$$(B_3) : \int_1 (y(t)^T g(t, \dots) + r(t)^T h(t, \dots)) \text{ is convex;}$$

$$(B_4) : \int_1 \lambda^T f(t, \dots) dt \text{ is quasicovex.}$$

$$(B_5) : \int_1 \lambda^T f(t, y, y', y'', y''') dt = \int_1 \bar{\lambda}^T f(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') dt$$

Then $y(t) = \bar{u}(t), t \in I$.

Proof : Proof is similar to 3.1.1.

4. WOLFE TYPE MULTIOBJECTIVE DUALITY

To establish duality results similar to the proceeding ones but under different convexity and generalized convexity assumption we formulate the following Wolfe type dual to the problem (P_λ) stated in the 2.1.1 Proposition.

We assume that $\bar{\lambda}$ is known and $\bar{\lambda} > 0$.

(WCD) $_{\lambda}$: Maximize :

$$\int_1 (\bar{\lambda}^T f(t, x, x', x'', x''') + s(t)^T g(t, x, x', x'', x''') + r(t)^T h(t, x, x', x'', x''')) dt$$

Subject to

$$x(a) = 0 = x(b), x'(a) = 0 = x'(b) \quad (11)$$

$$\begin{aligned} & \lambda^T f_u(t, u, u', u'', u''') + s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''') \\ & -D(\lambda^T f_{u'}(t, u, u', u'', u''') + s(t)^T g_{u'}(t, u, u', u'', u''') + r(t)^T h_{u'}(t, u, u', u'', u''')) \\ & +D^2(\lambda^T f_{u''}(t, u, u', u'', u''') + s(t)^T g_{u''}(t, u, u', u'', u''') + r(t)^T h_{u''}(t, u, u', u'', u''')) \\ & -D^3(\lambda^T f_{u'''}(t, u, u', u'', u''') + s(t)^T g_{u'''}(t, u, u', u'', u''') + r(t)^T h_{u'''}(t, u, u', u'', u''')) = 0 \end{aligned} \quad (12)$$

$$\text{and } \lambda > 0, \lambda \in \mathbb{R}^n, s(t) \geq 0, t \in I \quad (13)$$

In the following L_p represents the set of feasible solutions of (P_λ) and L_D the set of feasible solution of (WCD_λ) .

4.1 Theorem : Assume

$$(H_1) \mathbf{y}(t) \in L_p : (\bar{u}(t), \bar{s}(t), \bar{r}(t)) \in L_D$$

$$(H_2) \int_1^{\bar{T}} \bar{\lambda}^T f(t, \dots) \text{ and } \int_1^{\bar{T}} s(t)^T g(t, \dots) + r(t)^T h(t, \dots) dt \text{ are convex.}$$

Then,

$$\int_1^{\bar{T}} \bar{\lambda}^T f(t, x, x', x'', x''') dt \geq \int_1^{\bar{T}} \bar{\lambda}^T f(t, u, u', u'', u''') + s(t)^T g(t, u, u', u'', u''') + r(t)^T h(t, u, u', u'', u''')$$

Proof : By the convexity of $\int_1^{\bar{T}} \bar{\lambda}^T f(t, \dots) dt$, we have

$$\begin{aligned} \int_1^{\bar{T}} \bar{\lambda}^T f(t, x, x', x'', x''') dt & \geq \int_1^{\bar{T}} \bar{\lambda}^T f(t, u, u', u'', u''') dt + \int_1^{\bar{T}} [(x-u)^T \bar{\lambda}^T f_u(t, u, u', u'', u''') \\ & + (x'-u')^T \bar{\lambda}^T f_{u'}(t, u, u', u'', u''') + (x''-u'')^T \bar{\lambda}^T f_{u''}(t, u, u', u'', u''') + (x'''-u''')^T \bar{\lambda}^T f_{u'''}(t, u, u', u'', u''')] dt \\ & \dots \end{aligned} \quad (14)$$

From the dual constraints (12), we have,

$$\begin{aligned} \int_1^{\bar{T}} (x-u)^T & [(\lambda^T f_u(t, u, u', u'', u''') + s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''')) \\ & -D(\lambda^T f_{u'}(t, u, u', u'', u''') + s(t)^T g_{u'}(t, u, u', u'', u''') + r(t)^T h_{u'}(t, u, u', u'', u''')) \\ & +D^2(\lambda^T f_{u''}(t, u, u', u'', u''') + s(t)^T g_{u''}(t, u, u', u'', u''') + r(t)^T h_{u''}(t, u, u', u'', u''')) \\ & -D^3(\lambda^T f_{u'''}(t, u, u', u'', u''') + s(t)^T g_{u'''}(t, u, u', u'', u''') + r(t)^T h_{u'''}(t, u, u', u'', u'''))] dt = 0 \end{aligned}$$

This, by integrating by parts and using the boundary conditions as earlier, implies

$$\begin{aligned} & \int_1 (x-u)^T [(\lambda^T f_u(t, u, u', u'', u''') + s(t)^T g_u(t, u, u', u'', u''')) \\ & + r(t)^T h_u(t, u, u', u'', u''') + (x'-u')^T (\lambda^T f_{u'}(t, u, u', u'', u''')) \\ & + s(t)^T g_{u'}(t, u, u', u'', u''') + r(t)^T h_{u'}(t, u, u', u'', u''') + \\ & (x''-u'') (\lambda^T f_{u''}(t, u, u', u'', u''') + s(t)^T g_{u''}(t, u, u', u'', u''') + r(t)^T h_{u''}(t, u, u', u'', u'''))] dt = 0 \end{aligned}$$

Using this in (14) we have

$$\begin{aligned} & \int_1 \bar{\lambda}^T f(t, x, x', x'', x''') dt \geq \int_1 \bar{\lambda}^T f(t, u, u', u'', u''') dt \\ & - \int_1 (x-u)^T (s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''')) \\ & + \int_1 (x'-u')^T (s(t)^T g_{u'}(t, u, u', u'', u''') + r(t)^T h_{u'}(t, u, u', u'', u''')) \\ & + \int_1 (x''-u'')^T (s(t)^T g_{u''}(t, u, u', u'', u''') + r(t)^T h_{u''}(t, u, u', u'', u''')) \\ & + \int_1 (x'''-u''')^T (s(t)^T g_{u'''}(t, u, u', u'', u''') + r(t)^T h_{u'''}(t, u, u', u'', u''')) dt \end{aligned}$$

By hypothesis (H₂), this implies

$$\begin{aligned} & \int_1 \bar{\lambda}^T f(t, x, x', x'', x''') dt \geq \int_1 \bar{\lambda}^T f(t, u, u', u'', u''') dt \\ & + \int_1 (s(t)^T g(t, u, u', u'', u''') + r(t)^T h(t, u, u', u'', u''')) dt \\ & - \int_1 (s(t)^T g(t, u, u', u'', u''') + r(t)^T h(t, u, u', u'', u''')) dt \end{aligned}$$

Since $x \in L_p$, this implies

$$\int_1 \lambda^T f(t, x, x', x'', x''') dt \geq \int_1 (\lambda^T f(t, u, u', u'', u''') + s(t)^T g(t, u, u', u'', u''') + r(t)^T h(t, u, u', u'', u''')) dt$$

The following theorem gives a situation in which a Pareto optimal solution of (VP) exists

4.1.1 Theorem : Suppose

$$(F_1) : y(t) \in L_p : (\bar{u}(t), \bar{s}(t), \bar{r}(t)) \in L_D;$$

$$(F_2) : \int_1 \bar{\lambda}^T f(t, \dots) dt \text{ and } \int_1 [s(t)^T g(t, \dots) + r(t)^T h(t, \dots)] dt \text{ are convex.}$$

$(F_3) :$

$$\int_1 \bar{\lambda}^T f(t, x, x', x'', x''') dt = \int_1 \bar{\lambda}^T f(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + s(t)^T g(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + r(t)^T h(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') dt$$

Then $y(t)$ and $(\bar{s}(t), \bar{r}(t), \bar{u}(t))$ are optimal solutions of (P_λ) and (WCD_λ) . Hence $y(t)$ is a Pareto optimal solution of (VP).

Proof : Suppose $y(t)$ does not minimize (P), then there exists, $\bar{y}(t) \in L_p$ such that

$$\begin{aligned} \int_1 \bar{\lambda}^T f(t, \bar{y}(t), \bar{y}'(t), \bar{y}''(t), \bar{y}'''(t)) dt &< \int_1 \bar{\lambda}^T f(t, \bar{x}, \bar{x}', \bar{x}'', \bar{x}''') dt \\ &= \int_1 (\bar{\lambda}^T f(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + s(t)^T g(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + r(t)^T h(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) dt \end{aligned}$$

This contradicts the conclusion of 4.1. Theorem Hence $y(t)$ minimizes (P_λ) .

We can similarly prove that $(\bar{s}(t), \bar{r}(t), \bar{u}(t))$ maximizes (WCD_λ) and $y(t)$ is Pareto optimal solution of (VP), follows from proposition 2.1.

4.1.2 Theorem : Assume

$$(1) \bar{x}(t) \in L_p; (\bar{s}(t), \bar{r}(t), \bar{u}(t)) \in L_D$$

$$(2) \int_1 (\bar{\lambda}^T f(t, \dots) + \int_1 s(t)^T g(t, \dots) + r(t)^T h(t, \dots)) dt \text{ is convex.}$$

$$(3) \int_1 (\bar{\lambda}^T f(t, y, \bar{y}', \bar{y}'', \bar{y}''') dt = \int_1 (\bar{\lambda}^T f(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + s(t)^T g(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + r(t)^T h(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) dt$$

$$\text{Then } \int_1 s(t)^T g(t, \bar{x}, \bar{x}', \bar{x}'', \bar{x}''') dt = 0, t \in I$$

Proof : By hypothesis (2) and (3), we have

$$\begin{aligned} \int_1 \bar{\lambda}^T f(t, y, y', y'', y''') dt &= \int_1 (\bar{\lambda}^T f(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + s(t)^T g(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + r(t)^T h(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) dt \\ &\leq \int_1 (\bar{\lambda}^T f(t, y, y', y'', y''') + \bar{s}(t)^T g(t, y, y', y'', y''') + \bar{r}(t)^T h(t, y, y', y'', y''')) dt \\ &\quad - \int_1 \left[(y - \bar{u})^T (\bar{\lambda}^T f_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) + \right. \\ &\quad (y' - \bar{u}')^T (\bar{\lambda}^T f_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) + \\ &\quad (y'' - \bar{u}'')^T (\bar{\lambda}^T f_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) \\ &\quad \left. + (y''' - \bar{u}''')^T (\bar{\lambda}^T f_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) \right] dt \\ &= \int_1 (\bar{\lambda}^T f(t, y, y', y'', y''') + \bar{s}(t)^T g(t, y, y', y'', y''') + \bar{r}(t)^T h(t, y, y', y'', y''')) dt \\ &\quad - \int_1 \left[(y - \bar{u})^T (\bar{\lambda}^T f_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) + \right. \\ &\quad + \bar{r}(t)^T h_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') - (y - \bar{u})^T (\bar{\lambda}^T f_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) \\ &\quad \left. + \bar{s}(t)^T g_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right] \Bigg|_{t=a}^{t=b} \end{aligned}$$

$$\begin{aligned}
 & \int_1 (\bar{\lambda}^T f(t, y, y', y'', y''') + \bar{s}(t)^T g(t, y, y', y'', y''') + \bar{r}(t)^T h(t, y, y', y'', y''')) - \\
 & (y - \bar{u})^T \left(\bar{\lambda}^T f_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \Big|_{t=a}^{t=b} \\
 & - \int_1 \left[(y - \bar{u})^T D \left(\bar{\lambda}^T f_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) dt \right. \\
 & \left. + (y' - \bar{u}')^T \left(\bar{\lambda}^T f_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right] \Big|_{t=a}^{t=b} \\
 & - \int_1 \left[(y' - \bar{u}')^T D^2 \left(\bar{\lambda}^T f_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) dt \right. \\
 & \left. + (y'' - \bar{u}'')^T \left(\bar{\lambda}^T f_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right] dt \\
 & \left. + (y''' - \bar{u}''')^T \left(\bar{\lambda}^T f_{u''''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u''''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u''''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right] dt
 \end{aligned}$$

Integrating by parts,

$$\begin{aligned}
 & \int_1 (\bar{\lambda}^T f(t, y, y', y'', y''') + \bar{s}(t)^T g(t, y, y', y'', y''') + \bar{r}(t)^T h(t, y, y', y'', y''')) dt + \\
 & \int_1 \left[(y - \bar{u})^T \left(\bar{\lambda}^T f_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) - \right. \\
 & \left. (y - \bar{u})^T D \left(\bar{\lambda}^T f_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right] dt - \\
 & (y - \bar{u})^T D \left(\bar{\lambda}^T f_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \Big|_{t=a}^{t=b} \\
 & + \int_1 \left[(y - \bar{u})^T D^2 \left(\bar{\lambda}^T f_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right] dt
 \end{aligned}$$

(Using boundary conditions and integrating by parts)

$$\begin{aligned}
 &= \int_1 \left[(\bar{\lambda}^T f(t, y, y', y'', y''') + \bar{s}(t)^T g(t, y, y', y'', y''') + \bar{r}(t)^T h(t, y, y', y'', y''')) dt + \right. \\
 &\quad \int_1 \left[(y - \bar{u})^T (\bar{\lambda}^T f_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) \right. \\
 &\quad \left. D \left(\bar{\lambda}^T f_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right] dt - \\
 &\quad \left. D^2 \left(\bar{\lambda}^T f_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T g_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right] dt \\
 &= \int_1 \left(\bar{\lambda}^T f(t, y, y', y'', y''') + \bar{s}(t)^T g(t, y, y', y'', y''') + \bar{r}(t)^T h(t, y, y', y'', y''') \right) dt \quad (\text{Using (13)})
 \end{aligned}$$

$$\text{This implies } \int_1 \left(\bar{s}(t)^T g(t, y, y', y'', y''') + \bar{r}(t)^T h(t, y, y', y'', y''') \right) dt \geq 0 \quad (15)$$

But $\bar{s}(t) \geq 0, g(t, y, y', y'', y''') \leq 0$ and $h(t, y, y', y'', y''') dt = 0, t \in I$ yield

$$\int_1 \left(\bar{s}(t)^T g(t, y, y', y'', y''') + \bar{r}(t)^T h(t, y, y', y'', y''') \right) dt \leq 0 \quad (16)$$

From (15) and (16), we have

$$\int_1 \left(\bar{s}(t)^T g(t, y, y', y'', y''') + \bar{r}(t)^T h(t, y, y', y'', y''') \right) dt = 0$$

This, because of $h(t, y, y', y'', y''') = 0, t \in I$ gives $\int_1 \bar{s}(t)^T g(t, y, y', y'', y''') dt = 0$

This, together with $\bar{s}(t)^T g(t, y, y', y'', y''') \leq 0, t \in I$ implies

$$\bar{s}(t)^T g(t, y, y', y'', y''') = 0, t \in I$$

4.1.3 Theorem : Suppose

$$(1) \left(\bar{s}(t)^T, \bar{r}(t)^T u(t) \right) \in L_D \text{ and } \bar{u}(t) \in L_p$$

$$(2) \left(\bar{s}(t)^T g(t, u, \bar{u}', \bar{u}'', \bar{u}''') \right) = 0, t \in I$$

$$(3) \int_1 \bar{\lambda}^T f(t, \dots) dt \text{ and } \int_1 (\bar{s}(t)^T g(t, \dots) + \bar{r}(t)^T (t, \dots)) dt \text{ are convex;}$$

Then $\bar{u}(t)$ is an optimal solution of (P_λ) and hence of (VP).

Proof : If $\bar{u}(t)$ is only feasible solution of (P_λ) , then there is nothing to prove so, assume that $y(t)$ is another feasible solution of (P_λ) . Then by hypothesis (1) and (3), we have

$$\begin{aligned} \int_1 \bar{\lambda}^T f(t, y, y', y'', y''') dt &\geq \int_1 \bar{\lambda}^T f_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') dt \\ \int_1 \left[(y - \bar{u})^T (\bar{\lambda}^T f_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) + (y' - \bar{u}')^T (\bar{\lambda}^T f_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) \right. \\ &\quad \left. + (y'' - \bar{u}'')^T (\bar{\lambda}^T f_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) + (y''' - \bar{u}''')^T (\bar{\lambda}^T f_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) \right] dt \end{aligned}$$

Now integrating by parts, we have,

$$\begin{aligned} \int_1 \bar{\lambda}^T f(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') dt &- \int_1 [(y - \bar{u}) \bar{\lambda}^T f_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') dt \\ &- (y - \bar{u})^T (\bar{\lambda}^T f_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) \Big|_{t=a}^{t=b} - \int_1 (x - u)^T D (\bar{\lambda}^T f_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) dt \\ &+ (y' - \bar{u}')^T (\bar{\lambda}^T f_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) \Big|_{t=a}^{t=b} - \int_1 (y' - \bar{u}')^T D (\bar{\lambda}^T f_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) dt \\ &= \int_1 \bar{\lambda}^T f(t, u, \bar{u}', \bar{u}'', \bar{u}''') dt + \int_1 (y - \bar{u})^T D (\bar{\lambda}^T f(t, u, \bar{u}', \bar{u}'', \bar{u}''')) dt + \int_1 (y' - \bar{u}')^T D (\bar{\lambda}^T f_{u''}(t, u, \bar{u}', \bar{u}'', \bar{u}''')) dt \\ &\quad + \int_1 (y'' - \bar{u}'')^T D (\bar{\lambda}^T f_{u'''}(t, u, \bar{u}', \bar{u}'', \bar{u}''')) dt. \quad \text{(Using boundary conditions (11))} \end{aligned}$$

$$\begin{aligned} \int_1 \bar{\lambda}^T f(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') dt &+ \int_1 (y - \bar{u})^T \left[\bar{\lambda}^T f(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') - D (\bar{\lambda}^T f_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) \right. \\ &\quad \left. + D^2 (\bar{\lambda}^T f_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) + D^3 (\bar{\lambda}^T f_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''')) \right] dt \end{aligned}$$

$$\begin{aligned}
 &= \int_1 \left(\lambda^T f(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) dt \\
 &\quad - \int_1 (y - \bar{u})^T \left[\bar{s}(t)^T g_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right. \\
 &\quad \left. - D \left(\bar{s}(t)^T g_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right. \\
 &\quad \left. + D^2 \left(\bar{s}(t)^T g_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right. \\
 &\quad \left. - D^3 \left(\bar{s}(t)^T g_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right] dt
 \end{aligned}$$

Thus, by integrating by parts and using boundary conditions, as earlier, we get

$$\begin{aligned}
 &= \int_1 \bar{\lambda}^T f(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') dt - \int_1 \left[(y - \bar{u})^T \left(\bar{s}(t)^T g_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T h_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right. \\
 &\quad \left. + (y' - \bar{u}')^T \left(\bar{s}(t)^T g_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T h_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right. \\
 &\quad \left. + (y'' - \bar{u}'')^T \left(\bar{s}(t)^T g_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T h_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right. \\
 &\quad \left. + (y''' - \bar{u}''')^T \left(\bar{s}(t)^T g_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{s}(t)^T h_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) \right] dt \\
 &\geq \int_1 \left(\lambda^T f(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) dt + \int_1 \left[\bar{s}(t)^T g(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right] dt + \int_1 \bar{r}(t)^T h(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') dt \\
 &\quad - \int_1 \left[\bar{s}(t)^T g(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \bar{r}(t)^T h(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right] dt \\
 &\geq \int_1 \left(\lambda^T f(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \right) dt \quad \text{(Using hypothesis (A}_1\text{) and (A}_2\text{) and } x \in L_p\text{)}
 \end{aligned}$$

This implies that $\bar{u}$ minimizes $\int_1 \bar{\lambda}^T f(t, y, y', y'', y''') dt$ over L_p .

Remark : In 4.1.3 Theorem, we assume that a part of feasible solution of (WCD_λ) is a feasible solution of (P_λ) . It is a natural question if there is any set of appropriate conditions under which this assumption is true. The following theorem gives one such set of conditions.

4.1.4 Theorem : Assume

$$(1) x(t) \in L_p \text{ and } (\bar{s}(t)^T, \bar{r}(t)^T, u(t)) \in L_p$$

(2) $g(t, \dots)$ and $h(t, \dots)$ are differential convex functions.

$$(3) \int_I (g(t, u, u', u'', u''') + h(t, u, u', u'', u''')) dt = 0$$

$$(4) (x - u)^T g_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + (x' - u')^T g_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + \\ (x'' - u'')^T g_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + (x''' - u''')^T g_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \geq 0, t \in I$$

$$(5) (x - u)^T h_u(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + (x' - u')^T h_{u'}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \\ + (x'' - u'')^T h_{u''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') + (x''' - u''')^T h_{u'''}(t, \bar{u}, \bar{u}', \bar{u}'', \bar{u}''') \geq 0, t \in I$$

Then $\bar{u} \in L_p$.

Proof : By convexity of $g(t, \dots)$ and $h(t, \dots)$, we have

$$g(t, x, x', x'', x''') \geq g(t, u, u', u'', u''') + (x - u)^T g_u(t, u, u', u'', u''') \\ + (x' - u')^T g_{u'}(t, u, u', u'', u''') + (x'' - u'')^T g_{u''}(t, u, u', u'', u''') \\ + (x''' - u''')^T g_{u'''}(t, u, u', u'', u''') \geq 0 \quad \dots(17)$$

$$h(t, x, x', x'', x''') \geq h(t, u, u', u'', u''') + (x - u)^T h_u(t, u, u', u'', u''') \\ + (x' - u')^T h_{u'}(t, u, u', u'', u''') + (x'' - u'')^T h_{u''}(t, u, u', u'', u''') \\ + (x''' - u''')^T h_{u'''}(t, u, u', u'', u''') \geq 0 \quad \dots(18)$$

Using (13) and (14) together with the hypothesis (1), (2) and (3), we have

$$g(t, u, u', u'', u''') \leq 0, t \in I \quad (19)$$

and

$$h(t, u, u', u'', u''') \leq 0, t \in I \quad (20)$$

By (15) and (16), we have

$$g(t, u, u', u'', u''') + h(t, u, u', u'', u''') \leq 0, t \in I \quad (21)$$

The hypotheses (2) with (17) implies

$$g(t, u, u', u'', u''') + h(t, u, u', u'', u''') = 0, t \in I \quad (22)$$

But $g(t, u, u', u'', u''') \leq 0, t \in I$. Hence by (22) we have

$$h(t, u, u', u'', u''') \geq 0, t \in I \quad (23)$$

The inequalities (20) and (21) imply

$$h(t, u, u', u'', u''') = 0, t \in I \quad (24)$$

The relations (19) and (24) imply that $\bar{u} \in L_p$.

5. VARIATIONAL PROBLEMS WITH NATURAL BOUNDARY VALUES.

It is possible to construct variational problems with natural boundary values rather than the problems with fixed end points considered in the preceding sections. The problems of section 2 can be formulated as follows :

$$(\mathbf{VP})_N : \text{maximize} \left(\int_I f'(t, x, x', x'', x''') dt, \dots, \int_I f'(t, x, x', x'', x''') dt \right)$$

Subject to

$$g(t, x, x', x'', x''') \leq 0, t \in I$$

$$h(t, x, x', x'', x''') = 0, t \in I$$

$$(\mathbf{M} - \mathbf{WD})_N : \text{Maximize} \left(\int_I f'(t, u, u', u'', u''') dt, \dots, \int_I f'(t, u, u', u'', u''') dt \right)$$

$$(\lambda^T f_u(t, u, u', u'', u''')) + s(t) g_u(t, u, u', u'', u''') + r(t) h_u(t, u, u', u'', u''')$$

$$-D(\lambda^T f_u(t, u, u', u'', u''')) + s(t)^T g_{u'}(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''')$$

$$+ D^2(\lambda^T f_{u''}(t, u, u', u'', u''')) + s(t)^T g_{u''}(t, u, u', u'', u''') + r(t)^T h_{u'}(t, u, u', u'', u''')$$

$$- D^3(\lambda^T f_{u'''}(t, u, u', u'', u''')) + s(t)^T g_{u'''}(t, u, u', u'', u''') + r(t)^T h_{u''}(t, u, u', u'', u''') = 0$$

$$\int_I \left(s(t)^T g(t, u, u', u'', u''') + r(t)^T h(t, u, u', u'', u''') \right) dt \geq 0, \lambda > 0, s(t) \geq 0, t \in I$$

$$\lambda^T f_u(t, u, u', u'', u''') + s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''') = 0, t = a, t = b$$

$$\lambda^T f_{u''}(t, u, u', u'', u''') + s(t)^T g_{u''}(t, u, u', u'', u''') + r(t)^T h_{u''}(t, u, u', u'', u''') = 0, t = a, t = b$$

$$\lambda^T f_{u'''}(t, u, u', u'', u''') + s(t)^T g_{u'''}(t, u, u', u'', u''') + r(t)^T h_{u'''}(t, u, u', u'', u''') = 0, t = a, t = b$$

The above conditions are popularly known as natural boundary conditions in calculus of variation.

Theorem of the section 3 for $(VP)_N$ and $(M - WD)_N$ can easily be proved in view of earlier analysis in this research. The problems of the section 4 can be written with natural boundary values as follows :

For given $0 < \lambda < R^n$

$$(P_\lambda)_N : \text{Minimize } \int_I \lambda^T f(t, x, x', x'', x''') dt$$

Subject to

$$g(t, x, x', x'', x''') \leq 0, t \in I$$

$$h(t, x, x', x'', x''') = 0, t \in I$$

$$(WCD_\lambda)_N : \text{Maximize } \int_I (\lambda^T f(t, u, u', u'', u''') + s(t)^T g(t, u, u', u'', u''') + r(t)^T h(t, u, u', u'', u'''))$$

$$(\lambda^T f_u(t, u, u', u'', u''') + s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u'''))$$

$$-D(\lambda^T f_{u'}(t, u, u', u'', u''') + s(t)^T g_{u'}(t, u, u', u'', u''') + r(t)^T h_{u'}(t, u, u', u'', u'''))$$

$$+D^2(\lambda^T f_{u''}(t, u, u', u'', u''') + s(t)^T g_{u''}(t, u, u', u'', u''') + r(t)^T h_{u''}(t, u, u', u'', u'''))$$

$$-D^3(\lambda^T f_{u'''}(t, u, u', u'', u''') + s(t)^T g_{u'''}(t, u, u', u'', u''') + r(t)^T h_{u'''}(t, u, u', u'', u''')) = 0$$

$$\lambda^T f_u(t, u, u', u'', u''') + s(t)^T g_u(t, u, u', u'', u''') + r(t)^T h_u(t, u, u', u'', u''') = 0, t = a, t = b$$

$$\lambda^T f_{u'}(t, u, u', u'', u''') + s(t)^T g_{u'}(t, u, u', u'', u''') + r(t)^T h_{u'}(t, u, u', u'', u''') = 0, t = a, t = b$$

$$\lambda^T f_{u''}(t, u, u', u'', u''') + s(t)^T g_{u''}(t, u, u', u'', u''') + r(t)^T h_{u''}(t, u, u', u'', u''') = 0, t = a, t = b$$

$$\lambda > 0, s(t) \geq 0, t \in I.$$

CONCLUSION :

Variational Problems with natural boundary values rather than the problems with fixed end points are considered in different theorems. The results in this paper are generalization of those which are presented by I. Hussain, B. Ahmad and Z. Jabeen in the paper on Multiobjective Duality for variational problems. In this paper duality relationship between primal and its various duals are established.

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