$K_{p,q}$ graph - Edge Product Number

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Abstract

A graph G is said to be an edge product graph if there exists an edge function f: E $\rightarrow$ P such that the function f and its corresponding edge product function F on V satisfies that F(v)$\in$P for every v$\in$V and if e_1, e_2, …,$e_p$$\in$E such that $f(e_1).f(e_2).….f(e_p)$$\in$P then the edges e_1, e_2, …,e_p are incident on a vertex. In this paper the edge product number of complete bipartite graph is found.

Keywords: edge product function, edge product graph, edge product number of a graph

1. Introduction

We begin with simple, finite, connected and undirected graph G=(V(G),E(G)). For all other terminology and notations, follow Harary[1,2]. Here brief summary of definitions which are used for the present investigation are given. A dynamic survey on graph labeling is regularly updated by Gallian[3] and is published by electronic journal of combinatory.

2. Previous Results

Definition: 2.1

Let G be a graph and P be a set of positive integers with |E| = |P|. Then any bijection f: E $\rightarrow$ P is called an edge function of the graph G. Define F(v) = $\prod$\{f(e)/e is incident on v\}on V. Then F is called the edge product function of the edge function f. G is said to be an edge product graph if there exists an edge function f: E $\rightarrow$ P such that f and its corresponding edge product function F on V satisfy the conditions that F(v)$\in$P for every v$\in$V and if e_1,e_2, …,$e_p$$\in$E such that $f(e_1).f(e_2)$. … .$f(e_p)$$\in$P, then e_1, e_2, …,e_p are incident on a vertex.

Definition: 2.2

Let EPN(G), the edge product number of G, denote the minimum number K_2 components that must be added to G so that the resulting graph is an edge product graph. For any connected graph G other than K_2, EPN(G) $\geq$ 1. Let EPN(G) = r. An edge function f: E $\rightarrow$ P and its corresponding edge product function F which make G$\cup$rK_2 an edge product graph are respectively called an optimal edge function and an optimal edge product function of G.

Definition:2.3

Let $K_{(p, q)}$ be a complete bipartite graph with p+q vertices and pq edges. If G=$K_{(p, q)}$$\cup$r$K_2$ is a

graph with EPN($K_{p, q}$)=r where r ≥1 and 2≤p≤q. Let the vertex set and the edge set of $K_{(p, q)}$ be V_1={$u_1,u_2,…,u_p,v_1,v_2,…,v_q$} and E_1={u_iv_j:1≤i≤p and 1≤j≤q } respectively.

The mapping f:E→P is the optimal edge function and F is its corresponding optimal edge product function of f. The optimal edge function f: E→P can be restricted to the edge set E_1 and is represented by a matrix ($a_{i,j}$) of order (p×q), where $a_{i, j}$ =f(u_iv_j) for 1≤i≤p and 1≤j≤q. Then the matrix ($a_{i,j}$) is known as edge function matrix of the complete bipartite graph $K_{(p, q)}$. The row product and the column product of the optimal edge product function F is defined by

F(u_i) = i^{th} row product for 1 ≤ i ≤ p and

F(v_j) = j^{th} column product for 1 ≤ j ≤ q

Since u_i and v_j are the vertices for 1 ≤ i ≤ p and for 1 ≤ j ≤ q. The ith row elements are the labeling of the edges incident on u_i and the jth column elements are the labeling of the edges incident on v_j .

Definition: 2.4

Let $U_{(p,q)}$ be a matrix of order p×q. Then the two power unit product matrix $U_{(p, q)}$ is constructed by

$u_{(i, j)}$ = $2^{(i-1)+(j-1)(p-1)}$ for 1≤i≤(p-1) and for 1≤j≤(q-1)

$u_{(p ,j)}$ = $\left(\dfrac{1}{2^{(j-1)(p-1)} \prod\{2^{(k-1)} : 1 \le k \le (p-1)\}} \right)$ for 1≤j≤(q-1)]

$u_{(i, q)}$ = $\left(\dfrac{1}{2^{(i-1)} \prod\{2^{(k-1)(p-1)} : 1 \le k \le (q-1)\}} \right)$ for 1≤i≤(p - 1)]

$u_{(p, q)}$ = $\prod\{2^{k-1}: 1 \le k \le (p-1)(q-1)\}$

3. Main Result

Lemma: 3.1

If $U_{(p, q)}$ is a two power unit product matrix of order (p × q). Then

(i) The product of row elements and column elements of the matrix $U_{(p, q)}$ are equal to one.

(ii) The product of a collection of distinct elements of $U_{(p, q)}$ are one then they form a row or a column.

Theorem: 3.2

If $K_{(p, q)}$ is a complete bipartite graph then EPN($K_{p, q}$) ≥ 2 for 2≤p<q.

Proof:

Assume that for a complete bipartite graph $K_{(p, q)}$, EPN($K_{p, q}$)=1 for some 2≤p<q. Consider a graph G=$K_{(p, q)}$∪K_2 is an edge product graph for 2 ≤ p < q. Let V={$u_1, u_2 , … , u_p, v_1, v_2, …, v_q, w_1, w_2$} be the vertex set and E={u_iv_j: 1≤i≤p and 1≤j≤q}∪{w_1w_2} be the edge set of G. The mapping f: E → P is an optimal edge function and the function F is its corresponding optimal edge product function. Then F is an outer edge product function if $K_{(p, q)}$ has no pendent vertex and is triangle free. We have F(u_i) = F(v_j) = f(w_1w_2) = x(say), for 1≤i≤p and 1≤j≤q.

Hence $\prod\{\prod\{f(u_iv_j):1\le i\le p: 1\le j\le q\}$ = $\prod\{\prod\{f(u_iv_j): 1\le j\le q\}:1\le i\le p\}$.

Therefore, $\prod\{F(u_i):1\le i\le p\}$ = $\prod\{F(v_j):1\le j\le q\}$.

That is, $x^q = x^p \Rightarrow q = p$ which is a contradiction to our hypothesis that q > p. Hence we have EPN($K_{p, q}$)≥2 for 2≤p<q.

Example:

If p = 2, q = 4 and then take x = 2^5 we have the edge function matrix A of order (2×4) is:

$$\begin{bmatrix} 2^{12} & 2^{13} & 2^{14} & 2^5 \\ 2^{10} & 2^9 & 2^8 & 2^{17} \end{bmatrix}$$

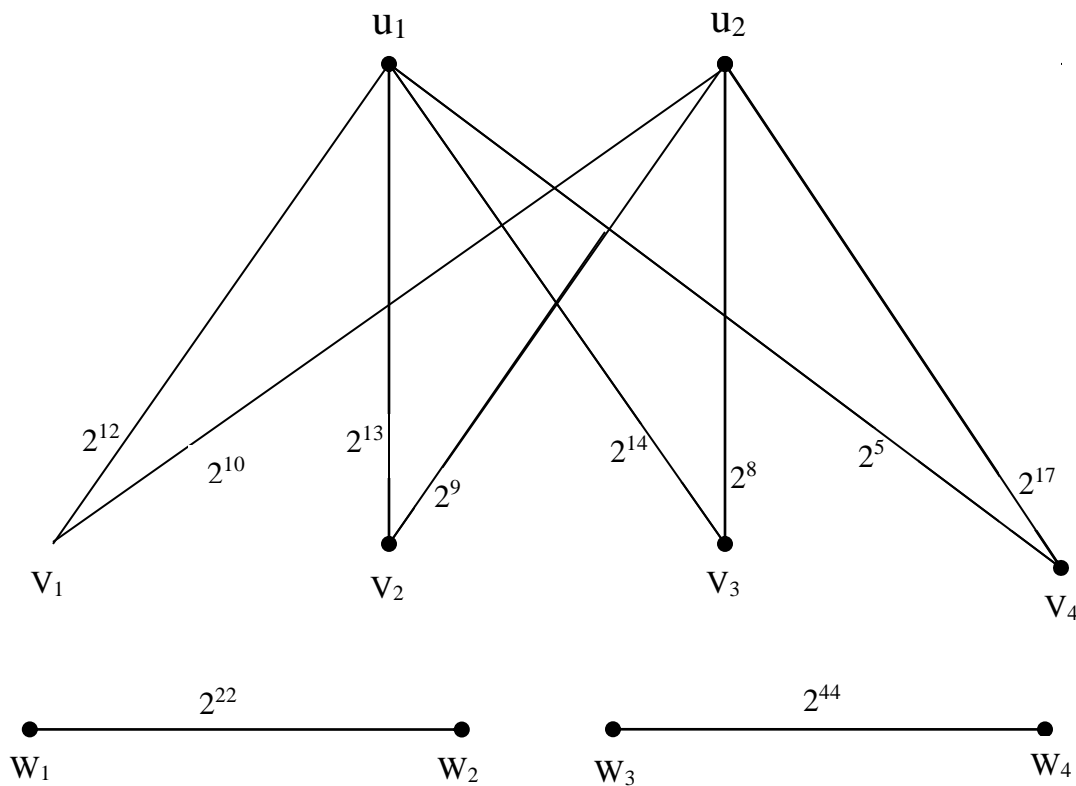The graph $K_{(2, 4)}∪2K_2$ is an edge product graph and EPN($K_{2, 4}$) = 2 is illustrated in the following figure.

Figure 1

Theorem: 3.3

If $K_{(p, q)}$ is a complete bipartite graph for $2 \le p < q$. Let $K_{(p, q)}∪2K_2$ be an edge product graph and EPN($K_{(p, q)}$) = 2. Let f: E→P be an optimal edge function and the function F be its corresponding optimal edge product function of the graph G and the range of F={x, y} then [g.c.d of {x, y}] $\notin$ {x, y}

Proof:

Consider a graph $K_{(p, q)}∪2K_2$ is an edge product graph for 2≤p<q with vertex set V={u_1, u_2, … , u_p, v_1, v_2, … , v_q, w_1, w_2, w_3, w_4} and edge set E={u_iv_j: 1≤i≤p and 1≤j≤q}∪{w_1w_2, w_3w_4}.

$$\text{Let } F\left(u_i\right) = F\left(v_j\right) = \begin{cases} x \text{ for } 1 \le i \le p_1 \text{ and} 1 \le j \le q_1 \\ y \text{ for } \left(p_1+1\right) \le i \le p \text{ and} \left(q_1+1\right) \le j \le q \end{cases}$$

If $p_2 = (p\,/\,p_1)$ and $q_2 = (q\,/\,q_1)$ then $\prod\{F(u_i){:}1{\le}i{\le}p\}= \prod\{F(v_j){:}1{\le}i{\le}q\}$

Therefore $x^{p1}y^{p2} = x^{q1}y^{q2}$. If $p_1{=}q_1$ then $q_2 = p_2$ and $p = p_1p_2 = q_1q_2 = q$ which is a contradiction to our hypothesis that q>p.

If $p_1{>}q_1$ then clearly $q_2 > p_2$. Consider [g.c.d of {x, y}] = y. Then y is a divisor of x.

Let $x = y^p$, then $y^{p(p1-q1)} = y^{(q2-p2)} \Rightarrow p = [(q_2/p_2)\,/\,(p_1/q_1)] \le (q_2/p_2) \le q_2.$

In one partition there are at least p vertices whose image under the function F which are equal to y, taking the entire edges incident on them we get that their total labeling is equal to x. But they are all not incident on a vertex.

This is a contradiction that the graph G is an edge product graph. Hence y is not a divisor of x. Similarly x is not a divisor of y.

Therefore [g.c.d of {x, y}] is neither x nor y. Thus we obtain that [g.c.d of {x, y}]$\notin${x, y}.

References

[1] F. Harary, Graph Theory, Addison - Wesley, Reading, 1969

[2] F. Harary, *Sum graphs over all the integers*, Discrete Math., 124 (1994) 99-105

[3] A. Gallian, *A dynamic survey of graph labeling,* The Electronic Journal of Combinatorics, 16 (2000) 39-48