Time dependent solution of Single Server feedback queue with Server is on Vacation

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Abstract:

In this paper we derive the time dependent solution of single server queue when the server is on vacation. The average queue length, various steady state probabilities are derived. The particular cases are also discussed. The numerical example illustrate that the efficiency of the results.

Key words: Transient probability, customer vacation, Feedback customer, Departure Process, Steay state probability.

Mathematical Subject Classification (2010): $60K25, 68M20, 90B22$.

1. Introduction

In the year of 1909, queueing theory originated in telephony with the work of Erlang[2]. After his work many authors to develop different types of queueing models, incorporating different arrival patterns, different service time distributions and various service disciplines. In the year of 1963, Takacs [12] first introduced queues with feedback mechanism which includes the possibility for a customer return to the counter for additional service. In the year of 1996, Gautam Choudhury [3] have proposed on a poisson queue with general setup time and vacation period. In the year of 1998, KrishnaReddy, Nadarajan and Arumuganathan [5] have investigated an M*/G(a,b)/1 queue with N-policy, multiple vacations and set-up times. In the year of 2000, Santhakumaran and Thangaraj [8] have studied a single server queue with impatient and feedback customers. In the year of 2008, Santhakumaran and Shanmugasundaram [9] have focused to study a Preparatory Work on Arrival Customers with a Single Server Feedback Queue

In queueing theory, the time independent solutions only derived for a long time. According to the theory and applications of queueing theory time dependent solution is necessary. Parthasarathy [7] and Parthasarathy and Sharafali [6] have discussed single and multiple server poisson queues of transient state solution in easiest manner. Krishna Kumar and Arivudainambi [4] has proposed a transient state solution for the mean queue size of M/M/1 queueing model when catastrophes occurred at the service station. Shanmugasundaram and Shanmugavadivu [10] have discussed a time dependent solution of single server queue with Markovian arrival and Markovian service. Shanmugasundaram and Chitra [11] have discussed time dependent solution of M/G2/1 retrial queue and feedback on Non Retrail customers with catastrophes. Kaliappan Kalidass and Kasturi Ramanath,have studied Transient Analysis of an M/M/1 Queue with Multiple Vacations. Chandrasekaran and Saravanarajan [1] has proposed a transient and reliability analysis of single server queue with feedback subject to catastrophes also discussed server failures and repairs. In this paper we analyze the time dependent solution of single server feedback queue with server vacation.

2. Model Description and Analysis

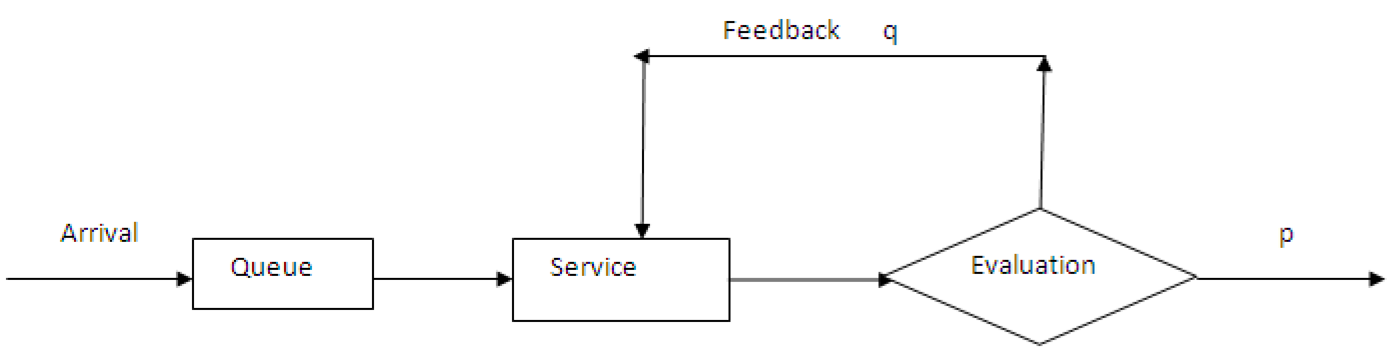


Figure 1 illustrates the flow of customers through the queueing system. External customers arrive according to a poisson process with rate λ .If the server is idle upon an arrival, service of an arriving customer starts instantaneously. After receiving service the customer a decision is made whether or not feedback. If a customer does feedback he joins the feedback stream with probability q and joins the end of the queue. The feedback is assumed to occur instantaneously. If a customer does not feedback, he joins the departure process with probability p and leaves the system forever. The queue discipline is FIFO and infinite in capacity, the service times are non-negative, independent and identically distributed random variable with parameter μ.

$$\mu = \begin{cases} \mu_1 & \text{when the customer get service with feedback} \\ \mu_2 & \text{when the customer get service without feedback} \end{cases}$$

The server completes all the services, he takes vacation. The rate of vacation time γ which is exponentially distributed random variable. The vacation time extends when no other customers in the queue. The motivation for this model is comes from Production system, Bank, Hospital, etc.

Let $p_n(0,t)$ denote the probability of n customers in the system at a time t when the server is on vacation. Let $p_n(1,t)$ denote the probability of n customers in the system at a time t when the server is available. Let $Q_1(z,t) = \sum_{n=1}^{\infty} p_n(1,t)\, z^n$ and $Q_0(z,t) = \sum_{n=1}^{\infty} p_n(0,t)\, z^n$ be the probability generating function for $|z| \le 1$. Generally we assume that the server is busy with i customers.

i.e. $$p_i(1,0) = \alpha_i \;\; i \ge 1 \;\; and \;\; p_i(0,0) = 0 \;\; i \ge 1 \tag{1}$$

where α_i be the probability of i customers in the system at time t = 0.

The system of differential – difference equations are

$$P'_0(0,t) = -\lambda P_0(0,t) + [p\mu_1 + q\mu_2] P_1(1,t) \tag{2}$$

For $n = 1,2,3,\ldots$

$$P'_n(0,t) = -(\lambda + \gamma) P_n(0,t) + \lambda P_{n-1}(0,t) \tag{3}$$

$$P'_1(1,t) = -[\lambda + p\mu_1 + q\mu_2] P_1(1,t) + \gamma P_1(0,t) + [p\mu_1 + q\mu_2] P_2(1,t) \tag{4}$$

For $n = 2,3,4,\ldots$

$$P'_n(1,t) = -[\lambda + p\mu_1 + q\mu_2] P_n(1,t) + \gamma P_n(0,t) + [p\mu_1 + q\mu_2] P_{n+1}(1,t) + \lambda P_{n-1}(1,t) \tag{5}$$

Taking Laplace Transform of equations (2) and (3) , we get

$$P_0^*(0,x) = \frac{1}{x+\lambda} + \frac{(p\mu_1 + q\mu_2)}{x+\lambda} P_1^*(1,x) \tag{6}$$

$$P_n^*(0,x) = \left(\frac{\lambda}{x+\lambda+\gamma}\right)^n P_{n-1}^*(0,x) \tag{7}$$

Recursively using equation (7), we get

$$P_n^*(0,x) = \left(\frac{\lambda}{x+\lambda+\gamma}\right)^n P_0^*(0,x) \quad (8)$$

The probability generating function $Q_1(z,t)$ satisfies the partial differential equation.

$$\frac{\partial Q_1(z,t)}{\partial t} = \left\{\lambda z + \frac{(p\mu_1+q\mu_2)}{z} - [\lambda + p\mu_1 + q\mu_2]\right\} Q_1(z,t) + \gamma Q_0(z,t) - (p\mu_1 + q\mu_2)P_1(1,t) - \gamma P_0(0,t) \quad (9)$$

Taking Laplace transform of equation (9) on both sides, we get,

$$\left[[x + \lambda + p\mu_1 + q\mu_2] - \left(\lambda z + \frac{(p\mu_1 + q\mu_2)}{z}\right)\right] Q_1^*(z,x)$$

$$= \alpha(z) + \gamma Q_0^*(z,x) - \gamma P_0^*(0,x) - (p\mu_1 + q\mu_2)P_1^*(1,x)$$

$$Q_1^*(z,x) = \frac{z\{\alpha(z)+\gamma Q_0^*(z,x)-\gamma P_0^*(0,x)-(p\mu_1+q\mu_2)P_1^*(1,x)\}}{[x+\lambda+p\mu_1+q\mu_2]z-\lambda z^2-(p\mu_1+q\mu_2)} \quad (10)$$

Equating the denominator as zero

$$\lambda z^2 - [x + \lambda + p\mu_1 + q\mu_2]z + (p\mu_1 + q\mu_2) = 0$$

The roots are

$$z_1 = \frac{w - \sqrt{w^2 - 4\lambda(p\mu_1 + q\mu_2)}}{2\lambda}$$

$$z_2 = \frac{w + \sqrt{w^2 - 4\lambda(p\mu_1 + q\mu_2)}}{2\lambda}$$

Where $w = x + \lambda + p\mu_1 + q\mu_2$ of which $|z_1| < 1$. Substitute $z = z_1$ in equation (10) we get,

$$\alpha(z_1) + \gamma Q_0^*(z_1,x) - \gamma P_0^*(0,x) - (p\mu_1 + q\mu_2)P_1^*(1,x) = 0$$

$$\alpha(z_1) + \gamma \frac{2(x + \lambda + \gamma)}{2(x + \lambda + \gamma) - (w - \sqrt{w^2 - 4\lambda(p\mu_1 + q\mu_2)}} P_0^*(0,x) - (x + \lambda + \gamma)P_0^*(0,x) = 0$$

Hence

$$P_0^*(0,x) = \frac{\alpha(z_1)}{x+\lambda+\gamma}\frac{2(x+\lambda+\gamma) - w - \sqrt{w^2 - 4\lambda(p\mu_1 + q\mu_2)}}{w + \sqrt{w^2 - 4\lambda(p\mu_1 + q\mu_2)} - 2(p\mu_1 + q\mu_2)}$$

$$= \alpha(z_1)F(x) \qquad (11)$$

Where $F(x) = \frac{w-\sqrt{w^2-4\lambda(p\mu_1+q\mu_2)}}{-2(p\mu_1+q\mu_2)x} + \frac{\left(w-\sqrt{w^2-4\lambda(p\mu_1+q\mu_2)}\right)^2}{4(p\mu_1+q\mu_2)x(x+\lambda+\gamma)} + \frac{1}{x} - \frac{w-\sqrt{w^2-4\lambda(p\mu_1+q\mu_2)}}{2x(x+\lambda+\gamma)}$

Taking inverse Laplace transform we get,

$$P_0(0,t) = \int_0^t \alpha(u)F(t-u)du \qquad (12)$$

Where $\alpha(t) = \sum_{n=1}^{\infty} \alpha_n I_n(2\sqrt{\lambda(p\mu_1 + q\mu_2)}t\,\frac{n}{t}\left(\sqrt{\frac{(p\mu_1+q\mu_2)}{\lambda}}\right)^n$

$$F(t) = 1 - \sqrt{\frac{\lambda}{(p\mu_1 + q\mu_2)}}\int_0^t I_1(2\sqrt{\lambda(p\mu_1 + q\mu_2)}\ u)\, e^{-[\lambda+(p\mu_1+q\mu_2)]u}du$$
$$+ \int_0^t \left(\frac{\lambda}{\lambda+\gamma} I_2\left(2\sqrt{\lambda(p\mu_1 + q\mu_2)}\,u\right) - \frac{\sqrt{\lambda}}{\lambda+\gamma} I_2\left(2\sqrt{\lambda(p\mu_1 + q\mu_2)}u\right)\right)(1$$
$$- e^{-(\lambda+\gamma)(t-u)})e^{-[\lambda+(p\mu_1+q\mu_2)]u}du$$

From equation (8) we get,

$$P_n(0,t) = \lambda^n \int_0^t P_0(0,u)\frac{(t-u)^{n-1}e^{-(\lambda+\gamma)(t-u)}}{(n-1)!}du \qquad (13)$$

$$P_1(1,t) = \frac{1}{\mu_t(p+q)}\left(P'_0(0,t) + \lambda P_0(0,t)\right) \qquad (14)$$

Taking inverse Laplace transform of equation (10) we get,

$$Q_1(z,t)$$
$$= \alpha(z) e^{[\lambda z + \frac{(p\mu_1+q\mu_2)}{z} - [\lambda+(p\mu_1+q\mu_2)]t} + \gamma \int_0^t Q_0(z,u)\, e^{[\lambda z + \frac{(p\mu_1+q\mu_2)}{z} - [\lambda+(p\mu_1+q\mu_2)](t-u)} du$$
$$- (p\mu_1 + q\mu_2) \int_0^t P_1(1,u) e^{[\lambda z + \frac{(p\mu_1+q\mu_2)}{z} - [\lambda+(p\mu_1+q\mu_2)](t-u)} du$$
$$- \gamma \int_0^t P_0(0,u)\, e^{[\lambda z + \frac{(p\mu_1+q\mu_2)}{z} - [\lambda+(p\mu_1+q\mu_2)](t-u)} du \qquad (15)$$

If $\theta = 2\sqrt{\lambda(p\mu_1 + q\mu_2)}$ and $\beta = \sqrt{\frac{\lambda}{(p\mu_1+q\mu_2)}}$ then $e^{\left[\lambda z + \frac{(p\mu_1+q\mu_2)}{z}\right]t} = \sum_{n=-\infty}^{\infty} I_n(\theta t)(\beta z)^n$

Where $I_n(.)$ is the modified Bessel function of the first kind.

Comparing the coefficients of z^n on both sides of (15) we get,

$$P_n(1,t) = \sum_{k=1}^{\infty} \alpha_k I_{n-k}(\theta t)\ \beta^{n-k} e^{-[\lambda+(p\mu_1+q\mu_2)]t}$$
$$+\ \gamma \int_0^t \sum_{k=0}^{\infty} P_k(0,u) I_{n-k}\big(\theta(t-u)\big)\ \beta^n\, e^{-[\lambda+(p\mu_1+q\mu_2)](t-u)} du$$
$$- \gamma \int_0^t P_0(0,u) I_n\big(\theta(t-u)\big)\, \beta^n\, e^{-[\lambda+(p\mu_1+q\mu_2)](t-u)} du - (p\mu_1 + q\mu_2)$$
$$\int_0^t P_1(1,u) I_n(\theta(t-u))\, \beta^n\, e^{-[\lambda+(p\mu_1+q\mu_2)](t-u)} du$$

The above equation can be written as

$$P_n(1,t) = \sum_{k=1}^{\infty} \alpha_k I_{n-k}(\theta t)\ \beta^{n-k} e^{-[\lambda+(p\mu_1+q\mu_2)]t} + \sum_{r=0}^{\infty} \alpha_{n-r} I_r(\theta t)\ \beta^{n-k} e^{-[\lambda+(p\mu_1+q\mu_2)]t}$$
$$+\ \gamma \int_0^t \sum_{k=0}^{\infty} P_k(0,u) I_{n-k}(\theta(t-u))\ \beta^n\, e^{-[\lambda+(p\mu_1+q\mu_2)](t-u)} du$$
$$+\gamma \int_0^t \sum_{r=0}^{\infty} P_{n-r}(0,u) I_r(\theta(t-u))\ \beta^{-r}\, e^{-[\lambda+(p\mu_1+q\mu_2)](t-u)} du$$
$$-\gamma \int_0^t P_0(0,u) I_n(\theta(t-u))\, \beta^n\, e^{-[\lambda+(p\mu_1+q\mu_2)](t-u)} du - (p\mu_1$$
$$+ q\mu_2) \int_0^t P_1(1,u) I_n(\theta(t-u))\, \beta^n\, e^{-[\lambda+(p\mu_1+q\mu_2)](t-u)} du \qquad (16)$$

Consider

$$\gamma \sum_{r=0}^{\infty} P_{n-r}{}^*(0,x) \left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda} \right)^r \frac{1}{\sqrt{w^2-\alpha^2}}$$
$$= \gamma \left(\frac{\lambda}{x+\lambda+\gamma} \right)^n P_0{}^*(0,x) \frac{1}{\sqrt{w^2-\alpha^2}} \sum_{r=0}^{\infty} \left(\frac{\lambda}{x+\lambda+\gamma} \right)^r \left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda} \right)^r$$

$$= \frac{\gamma}{2(p\mu_1+q\mu_2)x} \left(\frac{\lambda}{x+\lambda+\gamma} \right)^n \frac{[(w-\sqrt{w^2-\alpha^2}) - 2(p\mu_1+q\mu_2)]}{\sqrt{w^2-\alpha^2}} \sum_{k=0}^{\infty} \alpha_k \left(\frac{w-\sqrt{w^2-\alpha^2}}{\sqrt{w^2-\alpha^2}} \right)^k$$

Taking inverse Laplace transform of equation (17) on both sides and substitute in equation (16) we get,

$$P_n(1,t) = \sum_{r=1}^{n-1} \alpha_r I_{n-r}(\theta t)\ \beta^{n-r} e^{-[\lambda+(p\mu_1+q\mu_2)]t} + \sum_{r=0}^{\infty} \alpha_{n-r} I_r(\theta t)\ \beta^{-r} e^{-[\lambda+(p\mu_1+q\mu_2)]t}$$
$$+\ \gamma \int_0^t \sum_{k=0}^{n-1} P_k(0,u) I_{n-k}\big(\theta(t-u)\big)\ \beta^{n-k}\, e^{-[\lambda+(p\mu_1+q\mu_2)](t-u)} du$$

$$-\frac{\gamma}{\beta^2}\sum_{k=1}^{\infty}\int_0^t \alpha_k\chi_n(t-u)\theta^{k-1}I_{k-1}(\theta u)\, e^{-[\lambda+(p\mu_1+q\mu_2)](t-u)}du$$

$$+\frac{\gamma}{2\lambda}\sum_{k=1}^{\infty}\int_0^t \alpha_k\chi_n(t-u)\theta^{k}I_{k}(\theta u)\, e^{-[\lambda+(p\mu_1+q\mu_2)](t-u)}du-(p\mu_1$$

$$+q\mu_2)\int_0^t P_1(1,u)I_n(\theta(t-u))\,\beta^n\, e^{-[\lambda+(p\mu_1+q\mu_2)](t-u)}du \qquad (17)$$

Where $\chi_n(t)=\int_0^t \frac{u^{n-1}e^{-[\lambda+\gamma]u}}{(n-1)!}du$

Hence equations (12),(13),(14) and (17) completely determine the state probabilities of $P_0(0,t), P_n(0,t), P_1(1,t)\ and\ P_n(1,t)$.

3. Average Queue Length

In this section we derive average number of customers in the system at time t

$$E[X(t)]=g(t)=\sum_{n=1}^{\infty}n[P_n(0,t)+P_n(1,t)]$$

Then $g'(t)=\sum_{n=1}^{\infty}n[P'_n(0,t)+P_n{}'(1,t)]$

From equations (3), (4) & (5) we get

$$g'(t)=\lambda-(p\mu_1+q\mu_2)+(p\mu_1+q\mu_2)m(t) \qquad (18)$$

Where $m(t)=\sum_{n=0}^{\infty}P_n(0,t)$

Taking Laplace transform of m(t) we get

$$m^*(x)=P_0^*(0,x)+\lambda P_0^*(0,x)\frac{1}{x+\gamma}$$

Then by taking inverse Laplace transform, we get,

$$m(t)=P_0(0,t)+\lambda\int_0^t P_0(0,u)e^{-\gamma(t-u)}du$$

Now $g(t) = (\lambda - (p\mu_1 + q\mu_2))t + (p\mu_1 + q\mu_2)\int_0^t m(x)dx + m(0)$

$$E[X(t)] = [\lambda - (p\mu_1 + q\mu_2)]t + (p\mu_1 + q\mu_2)\int_0^t m(x)dx + \sum_{n=1}^{\infty} n\alpha_n \quad (19)$$

Particular Case

Case (i)

When $\mu_2 \to 0$ *and* $\mu_1 \to \mu$ i.e., there is a customer with feedback then the average queue length is

$$E[X(t)] = [\lambda - p\mu]t + p\mu\int_0^t m(x)dx + \sum_{n=1}^{\infty} n\alpha_n \quad (20)$$

Case (ii)

When $\mu_1 \to 0$ *and* $\mu_2 \to \mu$ i.e., there is a customer without feedback then the average queue length is

$$E[X(t)] = [\lambda - q\mu]t + q\mu\int_0^t m(x)dx + \sum_{n=1}^{\infty} n\alpha_n \quad (21)$$

Remark: When $q = 1$ in equation (20) and when $p = 1$ in equation (21), then these results coincides exactly with the paper [ ].

4. Steady State Probability

In this section we will discuss the steady state probabilities for the single server feedback queue with server vacation.

Multiply equation (11) by x and taking limit as $x \to 0$, using final value theorem of Laplace transform we get,

$$\lim_{x\to 0} xP_0^*(0,x) = \frac{\gamma}{\lambda+\gamma}(1-\rho) \text{ where } \rho = \frac{\lambda}{(p\mu_1 + q\mu_2)}$$

By Tauberian theorem

$$P_{0,0} = \frac{\gamma}{\lambda + \gamma}(1 - \rho)$$

Similarly

$$P_{0,n} = \left(\frac{\lambda}{\lambda + \gamma}\right)^n \frac{\gamma}{\lambda + \gamma}(1 - \rho) \;\; n \geq 1$$

$$P_{1,n} = \frac{\rho^n (1 - \rho)\gamma}{\lambda + \gamma - (p\mu_1 + q\mu_2)}\left[1 - \left(\frac{(p\mu_1 + q\mu_2)}{\lambda + \gamma}\right)^n\right] \quad , \quad n \geq 1$$

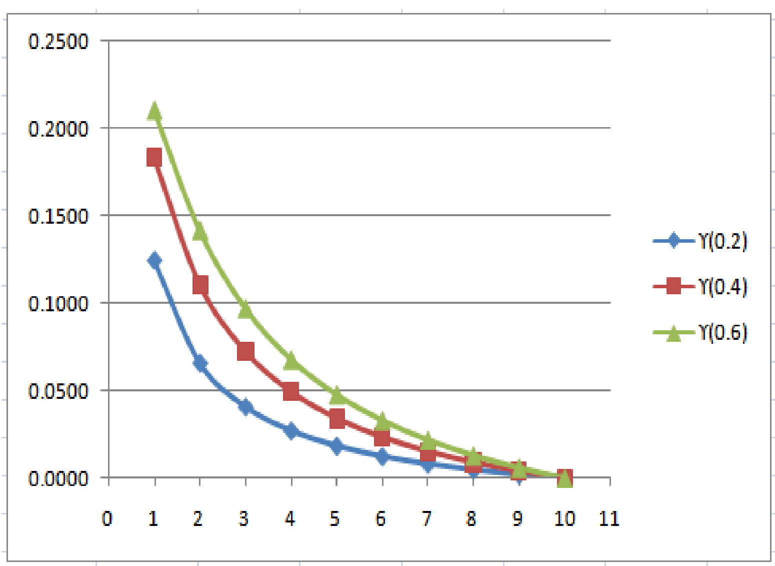
5. Numerical Analysis

In this section some numerical analysis along with its related graphs based on average queue length of the model are shown. The main intension is to illustrate the influence of the parameters p = 0.4 and q = 0.6 on the mean response time. For this purpose we have fixed vacation time $\gamma = 0.2, 0.4, 0.6$

Table :1

Computed value of $P_{0,0}$ for $\mu_1 = 10$ and $\mu_2 = 10$ with the fixed vacation time $\gamma = 0.2, 0.4, 0.6$

λ	ϒ(0.2)	ϒ(0.4)	ϒ(0.6)
1	0.1250	0.1837	0.2109
2	0.0661	0.1111	0.1420
3	0.0410	0.0727	0.0972
4	0.0272	0.0496	0.0681
5	0.0185	0.0343	0.0478
6	0.0125	0.0234	0.0331
7	0.0081	0.0153	0.0218
8	0.0048	0.0091	0.0130
9	0.0021	0.0041	0.0059
10	0.0000	0.0000	0.0000

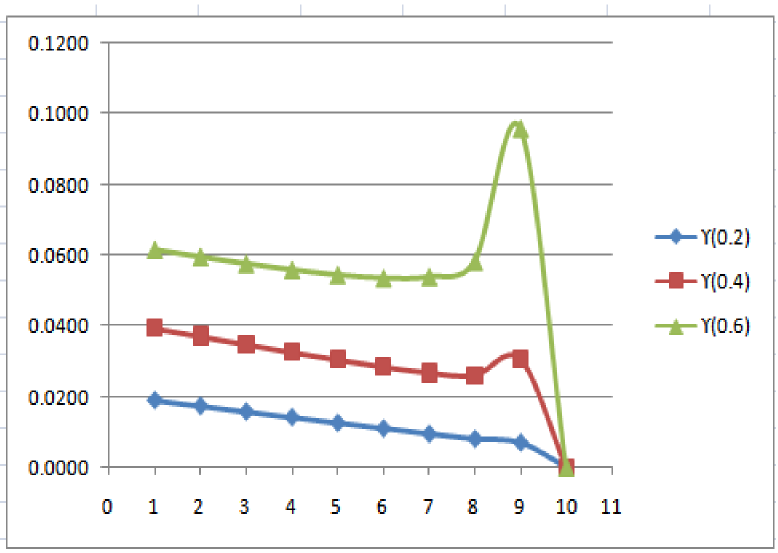


Arrival rate λ

Table :2

Computed value of $P_{1,n}$ for $\mu_1 = 10$ *and* $\mu_2 = 10$ with the fixed vacation time $\gamma = 0.2, 0.4, 0.6$.

λ	ϒ(0.2)	ϒ(0.4)	ϒ(0.6)
1	0.0188	0.0393	0.0617
2	0.0172	0.0371	0.0597
3	0.0156	0.0348	0.0578
4	0.0141	0.0326	0.0560
5	0.0125	0.0304	0.0545
6	0.0109	0.0284	0.0536
7	0.0094	0.0268	0.0540
8	0.0080	0.0260	0.0583
9	0.0070	0.0307	0.0960
10	0.0000	0.0000	0.0000



Arrival rate λ

6. Conclusion

The numerical example illustrate that when the server is on vacation, the average queue length is increased with the arrival rate. The vacation time increases with the increase of queue length. It shows the coincident.

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