

The boundedness of the solutions of degenerate divergent linear elliptic equations.

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Abstract: *We prove the boundedness estimates of solutions for degenerate elliptic-parabolic equations.*

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1 Introduction

Many applied problems is reduced to degenerate elliptic equations. For example, drift-diffusion processes in porous media, differently chemistry problems and so on. In papers [1]-[10] some correspondingly results is considered.

We classes degenerate linear elliptic equations is considered. Let in $Q_T = \Omega \times (0, T)$, where $\Omega \subset R^n$, $n \geq 2$ bounded domain with smooth boundary $\partial\Omega$, problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x, t) \frac{\partial u}{\partial x_j} \right) - \psi(x, t) \frac{\partial^2 u}{\partial t^2} + \\ + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u = 0, \end{aligned} \quad (1.1)$$

$$u(x, t) = f(x, t), \quad (x, t) \in \Gamma = (0, T) \times \partial\Omega \quad (1.2)$$

$$u(x, 0) = h(x), \quad x \in \Omega \quad (1.3)$$

is considered.

We consider the problem (1.1)-(1.3) under standard conditions for the functions $a_{ij}(x, t), b_i(x, t), c(x, t)$. Our main specific assumptions relayed with weight functions $\psi(x, t)$. About this conditions see [11].

A function $u(x, t) \in L_2(0, T; W_{2,\psi}^{1,2}(Q_T))$ is called of solution of problem (1.1)-(1.3) the integral identities

$$\begin{aligned} \int_{Q_T} \frac{\partial u}{\partial t} \bar{\varphi} dx dt + \int_{Q_T} \left[\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x, t) \frac{\partial u}{\partial x_j} \frac{\partial \bar{\varphi}}{\partial x_i} \right) + \right. \\ \left. + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} \bar{\varphi} + c(x, t)u \bar{\varphi} \right] dx dt + \end{aligned}$$

$$+ \int_{Q_T} \psi(x, t) \frac{\partial^2 u}{\partial t^2} \bar{\varphi} dx dt = 0 \quad (1.4)$$

hold for any functions $\bar{\varphi} \in C^\infty(\bar{Q}_T)$ vanishing near Γ and almost everywhere $\tau \in (0, T)$

$$u(x, t) - f(x, t) \in L_2(0, T; W_{2, \psi}^{\circ 1, 2}(Q_T)). \quad (1.5)$$

Theorem 1.1. *Let the standard conditions relayed the coefficients and weight functions for problem (1.1)-(1.3) be satisfied. Then there exists a constant M depending only on known parameters such that*

$$\text{ess sup}\{u(x, t) : (x, t) \in Q_T\} \leq M. \quad (1.6)$$

Proof. We shall prove estimates for u separately for the sets $\{u > 0\}$ and $\{u < 0\}$. We shall use the estimate

$$\begin{aligned} & \int_{\Omega} \left\{ \frac{1}{2} + [u - m_0]_+^3 \right\}^{r+1} dx + \\ & + \int_0^r \int_{\Omega} \omega^2(x) \Phi^{(r)}(u_k) \cdot \chi(m_0 < n < k) \cdot \left| \frac{\partial u}{\partial x} \right|^2 dx dt + \\ & + \int_0^\tau \int_{\Omega} \psi^2(x, t) \Phi^{(r)}(u_k) \chi(m_0 < n < k) \left| \frac{\partial^2 u}{\partial x^2} \right|^2 dx dt + \\ & + \frac{r+1}{k(r)} \int_0^r \int_{\Omega} (1 + |u|) [u_k - m_0] \Phi^{(r)}(u_k) dx dt. \end{aligned} \quad (1.7)$$

For get this estimate we substitute to integral identities (1.4) correspondingly test function, where

$$\Phi^{(r)}(u) = [u - m_0]_+ \left[\frac{1}{2} + [u - m_0]_+^3 \right]^r.$$

Let $\{\varphi_j(x)\}, j = 1, 2, \dots, J$, be a partition of unity such that

$$\sum_{j=1}^J \varphi_j(x) = 1, \left| \frac{\partial \varphi_j}{\partial x} \right| \leq \frac{C_0}{R} \quad \text{for } x \in \Omega,$$

$$\varphi_j(x) \in C^\infty(R^n), \text{supp} \varphi_j(x) \in B(x_j, R), J \leq \frac{C_0}{R^n}, r < 1.$$

where $B(x_j, R)$ is a ball of radius R with centre $x_j \in \Omega$, C_0 is a number depending only on n . The number R will be chosen later on. We test the integral identity (1.4) with the function

$$\bar{\varphi} = \sum_{j=1}^J u_k \Phi^{(r)}(u_k) [u - u_j] \varphi_j(x), u_j = u(x_j, t). \quad (1.8)$$

Integration with respect to t yields

$$\int_{Q_T} \omega(x) \Phi^{(r)}(u_k) \left| \frac{\partial u}{\partial x} \right|^2 dx dt = J_1 + J_2 + J_3$$

where

$$\begin{aligned} J_1 &= - \sum_{j=1}^J \sum_{i=1}^n \int_{Q_T} \omega(x) \Phi_1^{(r)}(u_k) |u - u_j| \varphi_j \frac{\partial u_k}{\partial x_i} \frac{\partial u}{\partial x_j} dx dt, \\ J_2 &= -2 \sum_{j=1}^J \sum_{i=1}^n \int_{Q_T} \omega(x) \Phi_1^{(r)}(u_k) |u - u_j| \varphi_j \frac{\partial \varphi_j}{\partial x_j} \frac{\partial u}{\partial x_j} dx dt, \\ J_3 &= - \sum_{j=1}^J \int_{Q_T} |u - f| u_k \Phi_1^{(r)}(u_k) |u - u_j| \varphi_j dx dt. \end{aligned}$$

For $\Phi_1^{(r)}(u_k)$ at $r \geq \frac{1}{2}$, $k > (m_0 + \frac{1}{2})$ have estimate

$$\Phi_1^{(r)}(u_k) \leq c_1(r+1) [u_k \Phi^{(r)}(u_k) \chi(m_0 < n < k)].$$

Later we get estimates for J_1 in case $\{u > 0\}$. We assume that $r \geq -\frac{1}{2}$ and choose the number R according to

$$R^\varepsilon = \frac{\varepsilon}{(r+1)^2}, \quad \varepsilon < \frac{1}{4}.$$

Then

$$\begin{aligned} |J_1| &\leq \varepsilon \left\{ \int_{Q_T} \omega(x) u_k \Phi^{(r)}(u_k) \left| \frac{\partial u}{\partial x} \right|^2 dx dt + \right. \\ &\quad \left. + C_2 \frac{1}{(r+1)^2} \int_{Q_T} \omega(x) u_k \Phi^{(r)}(u_k) \chi(m_0 < n < k) \left| \frac{\partial u}{\partial x} \right|^2 dx dt \right\}. \end{aligned}$$

By Cauchy inequality we have

$$|J_2| < \int_{Q_T} \omega(x) u_k \Phi^{(r)}(u_k) \left\{ \varepsilon_1 \left| \frac{\partial u}{\partial x} \right|^2 + \frac{1}{\varepsilon_1} \frac{C_3}{R^{n+2}} \right\} dx dt.$$

Correspondingly we have estimate J_3 . Applying the last estimate and choosing ε small enough we get

$$\begin{aligned} & \text{ess sup}_{\tau \in (0, T)} \int_{\Omega} \left\{ \frac{1}{2} + [u_k(\tau, x) - m_0]_+^3 \right\}^{r+1} dx + \\ & + \int_{Q_T} \omega^2(x) \Phi^{(r)}(u_k) \chi(m_0 < n < k) \left| \frac{\partial u}{\partial x} \right|^2 dx dt \leq \\ & \leq C_4(r+1)^{\lambda_1} \left\{ \int_{Q_T} \Phi^{(r)}(u_k) [u_k - m_0]^2 [u + |f|] dx dt + 1 \right\} \end{aligned}$$

with $\lambda_1 = 2(n+2) + 2$.

We want to apply Moser iteration with respect to the integral

$$I_k(r) = \int_{Q_T} \Phi^{(r)}(u_k) [u_k - m_0]^2 [u + |f|] dx dt. \quad (1.9)$$

To this end we use the embedding inequality

$$\begin{aligned} & \int_0^T \left\{ \int_{\Omega} |v(x, t)|^{2(1+\frac{2p}{n})} dx \right\}^{\frac{1}{p}} dt \leq \\ & \leq C_5 \left\{ \text{ess sup}_{t \in (0, T)} \int_{\Omega} v^2(x, t) dx \right\}^{\frac{1}{p} + \frac{2}{n} - 1} \int_{Q_T} \left| \frac{\partial v}{\partial x} \right|^2 dx dt \end{aligned} \quad (1.10)$$

which is fulfilled for $1 \leq p < \frac{n}{n-2}$ and arbitrary function $v \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W_{2, \omega}^1(\Omega))$. Applying Holders inequality to (1.9) we obtain

$$\begin{aligned} I_k(r) & \leq C_6 \int_0^T \left\{ \int_{\Omega} [\Phi^{(r)}(u_k) [u_k - m_0]^2]^p dx \right\}^{\frac{1}{p}} dt \leq \\ & C_7 \left\{ \text{ess sup}_{t \in (0, T)} \int_{\Omega} |\Phi^{(r)}(u_k)|^2 dx \right\}^{\frac{1}{p} + \frac{2}{n} - 1} \int_{Q_T} \left| \frac{\partial \Phi^{(r)}(u_k)}{\partial x} \right|^2 dx dt. \end{aligned} \quad (1.11)$$

We choose $r_j = \frac{1}{2}\theta^{-j} - 1, j = 0, 1, \dots$, and obtain from (1.11)

$$\{I_k(r_j)\}^{\theta-1} \leq C_7^{\theta-1} \theta^{-j} \{I_k(r_{j-1})\}^{\theta^{j-1}} \quad (1.12)$$

After iterating this estimate yields for arbitrary j the desired estimate. $\square$

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